ALGORITHM FOR THE AUTOMATIC DESIGN OF A SHADING DEVICE

by

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ABSTRACT

Given that there is a need to shade a window from the summer sun and also a need to expose it to the winter sun, this article describes an algorithm to design automatically a geometric construct that satisfies both requirements. The construct obtained represents the minimum solution to the simultaneous requirements. The window may be described by an arbitrary convex polygon and it may be oriented in any direction and it may be placed at any chosen latitude. The algorithm consists of two sequential steps: first to find a winter solar funnel surface; and the second to clip the surface subject to the summer shading conditions. The article introduces the design problem, illustrates the results through two examples, outlines the logic of the algorithm and includes the derivation of the mathematical relations required to implement the algorithm. This work is part of the MUSES project, which is a long term research effort to integrate Energy Consciousness with Computer Graphics in Architectural Design.

Nomenclature

A	a distance
a	a distance
b	unit vector binormal to a space curve. Completes triad (t, n, b) or (t, m, b)
c	a distance
d	a distance
l	unit vector tangent to a lateral edge of a solar funnel surface.
m	unit vector normal to solar funnel surface.
n	unit vector normal to window surface.
t	unit vector tangent to a window edge.
r	position vector of an arbitrary point.
r_1	position vector of a reference point.
q	position vector of a bounding edge corner.
S	a distance
S	vector distance from w to q.
w	position vector of a window corner.
σ	unit vector representing the direction of a solar ray.
σ_o	critical solar ray defining the tilt of the solar funnel surface.

Bold face letters represent vector quantities.
The vector product, also called the *cross* product, is represented with the symbol “x”.
The scalar product, also called the *dot* product, is represented with the symbol “◊”

INTRODUCTION

Consider, as an example, that we wish to design a direct solar gain house so as to provide passive solar heating during the winter months by exposing the windows to the sun while shading them during the summer in order to avoid overheating. The problem is to design fixed overhangs that satisfy both requirements. This design problem, of course, is not new. In fact it is addressed daily by designers concerned with this issue, and the solution for the required overhang is traditionally carried out manually. However, the manual solution can be carried out easily in the northern hemisphere only for south facing windows and in the southern hemisphere only for north facing windows. The solution for the overhangs for windows oriented differently becomes sufficiently laborious so that it is often avoided. In such cases easier solutions are adopted, such as using moveable shades or even excluding windows altogether.

Fixed overhangs designed for conditions of winter exposure and summer shading are formally interesting and it may be useful to be able to investigate them for possible integration into the overall design provided, of course, such investigation could be done quickly. To this end I wrote the computer program that would permit this investigation, and in this article I describe the overall logic and the key mathematical solutions at the core of the program.

The program, named SHADESIGN, uses the same interface as well as the underlying data structure as the rest of MUSES . It can export and import data with other modules of MUSES or it can run in the stand alone mode.

Fig. 1 illustrates the user interface of the program. It shows the solutions for the "overhangs" for two "arched" windows, one facing south and the other facing east at latitude 30.5 ° N. Each output consists of four views. Three of them are coupled orthographic views: starting from the bottom right and advancing counter clockwise we find respectively the plan, and then successively the south and east elevations. The fourth view, bottom left, is used for arbitrary oblique paraline views. In this example it shows the solar view at 9:00 a.m. during the winter design day (Nov. 25 and Jan. 15). The additional viewports on the right side of the Fig. 1 are used for entering data in a "dial" format. The top "dial" is used to select the winter and summer design days. In this example the windows are to be exposed to the sun 100% until January 16 and they are to be shaded 100% until August 6. Thus February 25 becomes the "winter design day" and August 31 becomes the "summer design day". Because of the solar symmetry about the solstices' axis these choices imply that the windows will be 100% exposed from November 25 through January 16 and 100% shaded from May 8 through August 6.

The remaining times are periods of transition with partial shading. The bottom viewport on the right is the window orientation dial which in this Fig. 1 are “east” and “south” for the respective examples.

Both overhang solutions display south seeking openings. In the case of the south facing window this is evident by the symmetry and in the case of the east facing window by the asymmetry. The outside edge of the overhang defines the minimum shading edge that satisfies both winter and summer design conditions. The nature of the solution adopted for the program reported in this article connects these outer edges to the window edges and corners. However, the edges may be connected in any other way to the wall associated with the window.

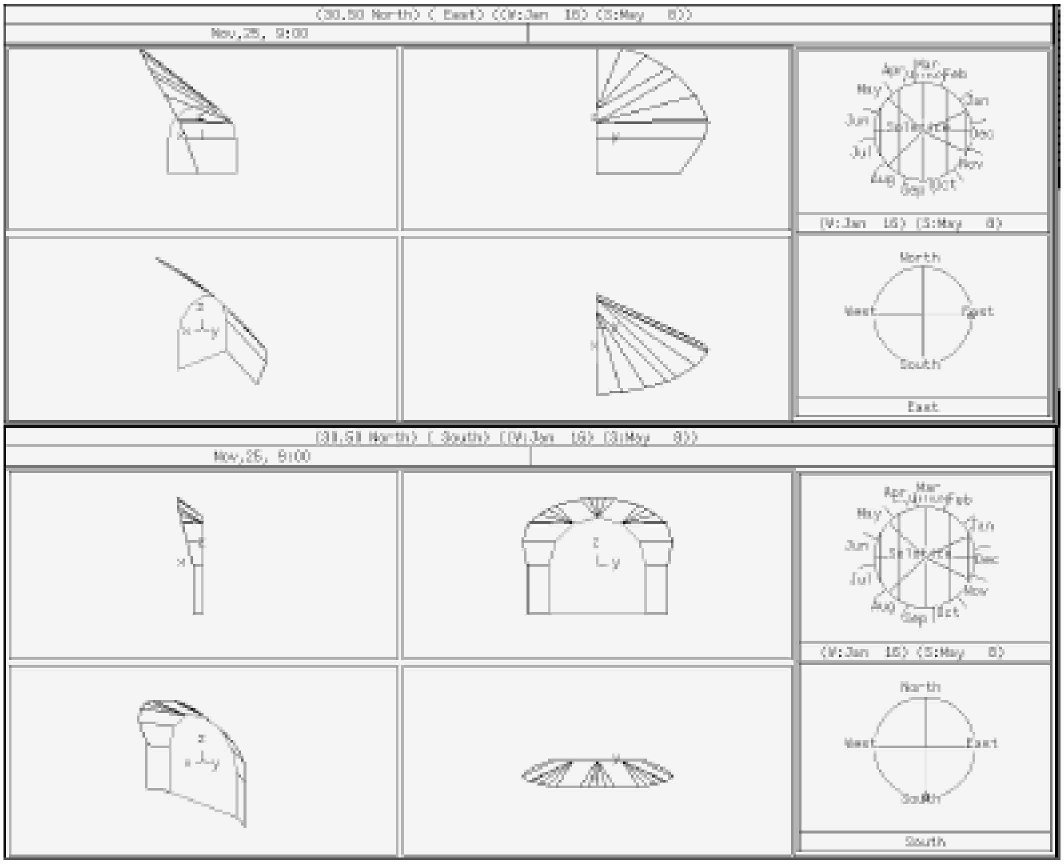

Fig. 1: The interface of the SHADESIGN program with two window solutions.

STRATEGY

Given the winter and summer design days the procedure to design the fixed overhang consists of two sequential steps: first, use the winter solar tilt to generate a *solar funnel* ; second, use the summer solar tilt to clip the solar funnel. The edge of the clipped funnel defines the minimum shading edge that satisfies both winter and summer requirements.

Defining the Winter Solar Funnel

The "solar funnel " is a curved developable space surface composed of a series of plane surfaces each of which is associated uniquely with either a window edge or a corner. Each surface is critically tilted relative to the window plane so that the window is never shaded during the winter design day.

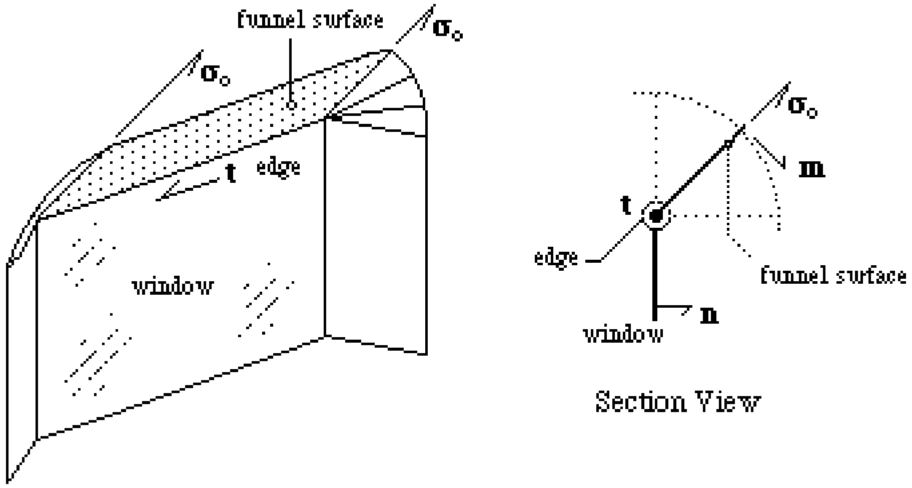

Fig. 2. The window and its associated developable funnel surface.

Fig. 2 shows a window with its solar funnel. The top window edge and its associated funnel surface are highlighted. The window orientation is defined by its unit normal vector "n" and the orientation of the edge by the unit vector "t". The section view of the window makes it easier to visualize a plane pivoted around the edge as if hinged. This plane is rotated until we find the minimum "tilt" that guarantees that the window can not be shaded by the plane during the winter design day. The "tilt" is defined by the unit vector "σ_o" which represents the position of the sun at the *critical time* of the day. The critical time, of course, is the time when the sun is in the direction "σ_o", thus knowledge of the critical time implies knowledge of "σ_o", and vice versa. We must remember that there is a different critical time for each funnel surface. The plane is bound by two lines directed along "σ_o" that go through either end of the window edge. I will refer to these lines as the lateral edges of the funnel surface. This plane is also bound by the window edge itself; consequently, it takes a quadrilateral form. The direction vector normal to the funnel surface is given by "m".

This procedure is repeated for each window edge. Each edge has a quadrilateral funnel surface associated with it and, of course, a "critical" time and its associated solar direction "σ_o". The spaces between adjacent quadrilateral funnel surfaces are filled with triangular funnel surfaces emanating from the window corner common to the respective window edges. The lateral edges of these triangular surfaces are given by "σ's" calculated at times of the day interpolated between the critical times of the two adjacent quadrilateral planes.

The lateral edges of each plane are effectively the “generators” of the funnel surface and since they are straight lines we can conclude that the solar funnel is a developable surface.

Summer Clipping of the Solar Funnel

For the summer design day we clip each edge of every funnel surface with summer rays emanating from each window corner until each lateral funnel edge is as short as possible.

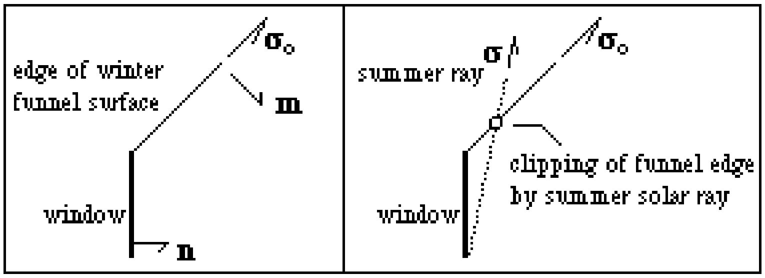

Fig. 3. Summer clipping of funnel surface.

Consecutive clip points then define sequential shading edges which represents the desired solution to the problem.

Mathematical Solution

This section derives the mathematical conditions to find the winter critical time for each quadrilateral solar funnel surface and the summer clipping position along each lateral edge. The results are summarized first and then they are derived in the subsections below.

The critical time “σ_o” that defines the tilt of the funnel surface associated with the window edge “t” is given by the condition “$t\Diamond(\sigma_o \text{ x } \sigma) < 0$”. “$\sigma$” is the solar position for any other time of the day other than the critical time.

A lateral edge directed along “l” is associated with a window corner “q” (see Fig. 5). If “S” is the position of “q” relative to the position of any other window corner “w” then during the summer design day there is only one time when a solar ray will go through “q” and also intersect the lateral edge. This time is given by the solar ray “σ” which satisfies the equation “$\sigma\Diamond(l \text{ x } S) = 0$”. This solar ray intersects the lateral edge at a distance “d” given by “$d = \sigma \Diamond(S \text{ x } b) / (\sigma\Diamond m)$”.

Fig. 4 illustrates the top view of the window highlighting a window edge which is considered to come out of the page. In this view the edge appears as a point.

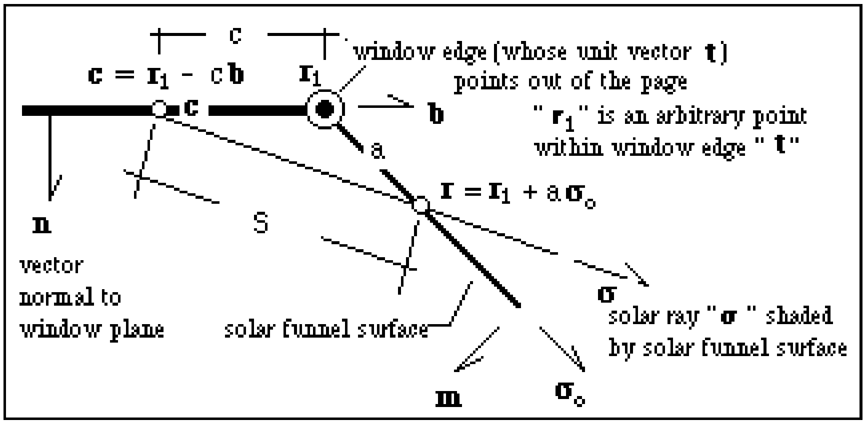

Fig. 4 Conditions to calculate the tilt of a winter funnel surface

Emanating from this edge, along the solar direction “σ_o”, is the surface of the solar funnel associated with this edge. “m” is the vector normal to this surface and it’s constructed such that the window plane is always in view of the solar funnel, i.e. “$n \Diamond m > 0$”. The vectors “σ_o”, “t” and “m” form an orthonormal triad “$m = \sigma_o \times t$ “.

Let “r” be an arbitrary point affixed on the solar funnel surface constructed such that

$$r = r_1 + a\, \sigma_o \qquad (1)$$

where “a” is any positive number and “r_1” is any point within the window edge.
For a given time during the winter design day, provided the sun is in front of the window “$\sigma \Diamond n > 0$”, “c” is the shadow cast by “r” on the window. “σ” is the corresponding position of the sun. The shadow point “c” must satisfy the following conditions

the point “c” rests on the window plane

$$(c - r_1) \Diamond n = 0 \qquad (2)$$

segment from “c” to “r” follows the sun

$$(r - c) \times \sigma = 0 \Rightarrow c = r + S\sigma \qquad (3)$$

segment from “r_1” to “c” follows the binormal

$$(r_1 - c) \times b = 0 \Rightarrow c = r_1 - cb \qquad (4)$$

The point “c” is a shadow on the window whenever the distance “c” is positive. The 2 line equations must yield the same “c”, therefore equating (3) to (4) and using (1)

$$c = c \quad \Rightarrow \quad r + S\sigma = r_1 - cb \quad \Rightarrow \quad c = -a\, \sigma_o \Diamond b - S\, \sigma \Diamond b \qquad (5)$$

From the plane equation (2) we can get an expression for "S" by using successively (3) and then (1)

$$(c - r_1)\Diamond n = 0 \quad \Rightarrow \quad (a\, \sigma_o + S\sigma)\Diamond n = 0 \quad \Rightarrow \quad S = -a\, \sigma_o \Diamond n \,/\, \sigma \Diamond n \qquad (6)$$

which is used to replace "S" in (5), then with some rearranging the distance "c" is given explicitly by

$$c = a\, t\Diamond(\sigma_o \times \sigma) \,/\, (\sigma \Diamond n) \quad (\sigma \Diamond n \neq 0) \qquad (7)$$

Since both "a" and "$(\sigma\Diamond n)$" are positive, then "c" will be positive or negative according to "$t\Diamond(\sigma_o \times \sigma)$" being positive or negative. The point "c" is a shadow of "r" provided "$c > 0$".

The objective of constructing the funnel is to guaranty that no point "r" on the funnel surface will ever cast a shadow on the window during the winter design day. Therefore we are looking for the condition " $c < 0$ " or

$$t\Diamond(\sigma_o \times \sigma) < 0 \qquad (8)$$

The critical time ($t \Rightarrow \sigma_o$) is obtained by searching through the winter design day for that one time such that for all other times condition (8) is satisfied.

Summer Clipping Of A Lateral Edge

First we find the time during the summer design day that the solar direction "σ" intersects both the window corner "w" and the lateral edge that goes through point "q" in the direction "l". The boundary edge has a binormal vector "b" in association with the funnel surface whose normal direction is "m" such that "$b = l \times m$". See Fig. 5.
The intersection point is "r" such that

$$(r - w) \times \sigma = 0 \quad \Rightarrow \quad r = w + A\sigma \qquad (9)$$

" r " must belong to two planes simultaneously, each one yielding a solution for the distance "A"

$$(r - q)\Diamond b = 0 \Rightarrow \quad A = S\Diamond b \,/\, \sigma\Diamond b \qquad (10)$$

and

$$(r - q)\Diamond m = 0 \Rightarrow \quad A = S\Diamond m \,/\, \sigma\Diamond m \qquad (11)$$

And of course the two separate solutions for "A" must be equal, thus equating (10) to (11) and after some rearranging

$$A = A \qquad \Rightarrow \qquad S\Diamond b \,/\, \sigma\Diamond b = S\Diamond m \,/\, \sigma\Diamond m \Rightarrow \sigma\Diamond(l \;x\; S) = 0 \qquad (12)$$

Which gives explicitly the time of the day when the summer solar ray intersects both the window corner “w” and the lateral edge of the funnel surface. Note that the point “ $r = w + A\sigma$ “ also satisfies the equation for the line of the bounding segment “$(r - q)\; x\; l = 0$” as it should.Second we find the distance “d” from the corner “q” to the intersection “r” is

$$d = (\, r - q\,)\Diamond l \qquad (13)$$

and after substituting equations (9) and (11) and after some rearranging

$$d = (\, A\sigma - S\,)\Diamond l \Rightarrow d = \sigma \Diamond(S \;x\; b) \,/\, (\sigma\Diamond m) \qquad (\sigma\Diamond m \neq 0) \qquad (14)$$

Mathematical Notes

Vector Identities

The following vector identities are useful in the “re arranging” of terms in the derivations given above.

$a \;x\; (\, b \;x\; c) = (a\Diamond c)\, b - (a\Diamond b)\, c$ and $a\Diamond(\, b \;x\; c\,) = b\Diamond(\, c \;x\; a\,) = c\Diamond(\, a \;x\; b\,)$

Local coordinate triads

A local coordinate system can always be assigned to any curve in space. In this article all curves are expressed as a sequence of straight edges. In Fig. 6, for example, the edge associated with a polygon has a triad of unit base vectors: “t”, the tangent to the edge; “n”, the vector normal to the polygon and “b” the binormal vector. By construction t, n and b are mutually perpendicular.

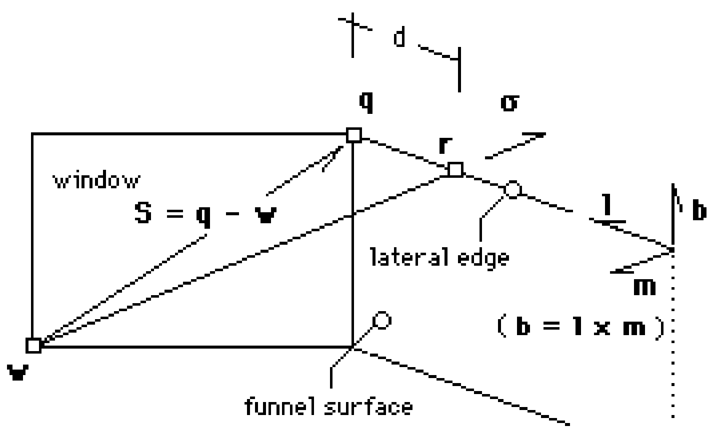

Fig. 5. Summer Clipping of of a Winter Funnel Surface

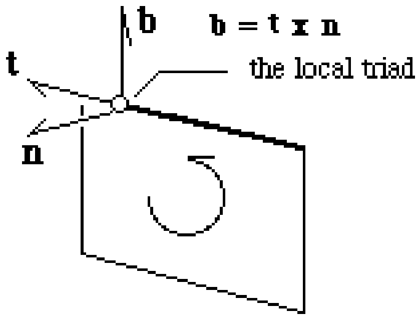

Fig. 6 The local triad of an edge of a polygon.

Solar coordinates

To obtain the solar position vector (σ) we need to specify: a) the latitude "L", positive values are for the north latitudes and negative values are for the south latitudes; b) the day of the year in a Julian count "J", e.g. Jan. 1 is "J = 1", Dec. 31 is "J = 365", June solstice is "J = 172.5"; and c) the solar time of the day "t". The Julian day is used to calculate the solar declination "g" and the solar time is used to calculate the diurnal angle "h".

$$g = 23.5^\circ \cos[\,360^\circ(J - 172.5)/365] \qquad (J = \text{Julian day from 1 to 365})$$

$$h = 15^\circ\, t \qquad (t = \text{solar time from 1 to 24})$$

The solar position vector expressed in terms of the cardinal triad is "$\sigma = (\sigma_s, \sigma_e, \sigma_z)$ " where

$$\sigma_s = \sin(L)\cos(g)\cos(h) - \cos(L)\sin(g) \qquad \text{(south)}$$

$$\sigma_e = -\sin(h)\cos(g) \qquad \text{(east)}$$

$$\sigma_z = \cos(L)\cos(g)\cos(h) + \sin(L)\sin(g) \qquad \text{(zenith)}$$

CONCLUSION

This article presents the logic and demonstrates the results for an algorithm to design automatically a fixed shading device for a window such that the window is exposed to the sun during a prescribed period in the winter season and it is shaded during a prescribed period during the summer season. The window may be described by an arbitrary convex polygon, it may be oriented along an arbitrary orientation and it may be located any latitude. The prescribed periods are defined by choosing two design days, one for winter and one for summer. The algorithm first finds a solar funnel which is a developable surface composed of plane polygons each one is associated either with a window edge or a window corner. The orientation of each plane polygon is derived from the winter solar ray relative to the edges of window. The second part of the algorithm cuts the funnel surface by clipping the lateral edges of the plane polygons with the summer solar rays. The locus of points defined by this clipping action define the minimum shading edge that satisfy both winter and summer requirements.