Four Algebraic Structures In Design

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Abstract
A constructive program for the generation of three-dimensional languages of designs based on nested group structures is outlined.

Keywords
Computational Design, Symmetry, Group Theory, Shape Grammars, VRML

1 Introduction

Dihedral and cyclic symmetries have been heavily used for the interpretation or the generation of two-dimensional designs in architectural composition and the arts in general. Several plans, elevations and sections in architectural design can be understood in terms of their relation to one or more underlying cyclic or dihedral group structures. Alternatively music scores exhibiting canonic or fugal writing in music composition can be understood in terms of their relation to specific underlying cyclic or dihedral group structures. Designs in these spatial and sound systems generally do not necessarily reveal immediately their underlying structures; a complex array of relations and ordering schemes are typically introduced to relate several parts to one another and to the overall configuration to interpret existing designs or produce new designs from scratch.

The four algebraic structures that extend the cyclic and dihedral symmetries of the Euclidean plane to the Euclidean space are discussed in this paper. All four structures, the cyclic group of order n, C_n, the dihedral group of order n, D_n, the direct product of a cyclic group of order n and a cyclic group of order 2, C_nxC_2, and the direct product of a dihedral group of order n and a cyclic group of order 2, D_nxC_2, are correlated for a specific order of 8 so that differences in their structures may be clearly illustrated and brought forward. These four instances, C_8, D_4, C_4xC_2, D_2xC_2, are represented in terms of partial order lattices and correlated with the 7 possible geometric structures of prismatic symmetry. It is suggested that the same lattices can be used for the computation of designs with no global symmetry but with a wealth of rich spatial connections among their parts. A simple case study in the end exemplifies this notion of generative applications of lattices and nested underlying structures in design.

2 The algebraic structure of the symmetry groups

A symmetry of a shape is an isometric transformation that leaves a shape invariant. The sum of all such isometric transformations that leave a shape invariant provides the degree or order of symmetry of the shape. However, simply counting symmetries is not enough; it has to be taken into consideration how they combine with one another. Consider for example the four shapes in figure 1. The first shape is a shaved octagonal truncated pyramid, that is, an octagonal pyramid has been given a clockwise or a counterclockwise orientation by shaving off its corner edges; the second shape is an isosceles tetrahedron, that is, a shape that can be obtained from a regular tetrahedron by expansion or compression along one of the axes that pass through the middle of two opposite edges; the third shape is a shaved square prism, that is, a square prism that has been given a clockwise or a counterclockwise orientation by truncating its corner edges; the fourth shape is a rectangular prism. All four shapes have the same order of symmetry equal to 8, but clearly, they do not exhibit the same structure. They look and feel very different. Various methods of transformations of existing polyhedra to other polyhedra with the same symmetry include techniques such as trun-

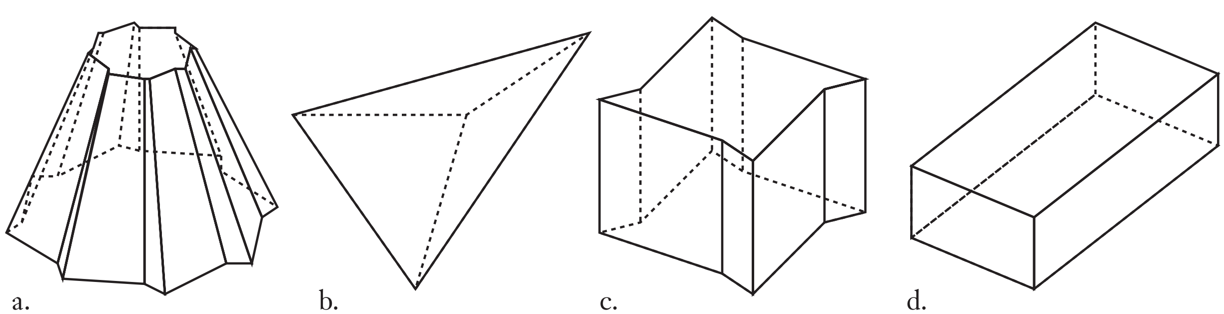


Figure 1. Four shapes with equal order of symmetry (a) Shaved octagonal truncated pyramid (b) Isosceles tetrahedron (c) Shaved square prism (d) Rectangular prism.

cation, distortion, dissection, folding, vertex motion and others (see, for example, Williams, 1979).

The depiction of the symmetry transformations of the four shapes makes explicit their differences. First, the first and the third shape possesses only one axis of rotational symmetry, as opposed to the second and the fourth shape that has three; the axis of rotational symmetry in these shapes induces a 4-fold rotation as opposed to the rotational axes of the other two shapes that induce half-turns. Moreover, in the first shape a rotor reflection of p/8 about its central axis produces all 8 symmetries when it is combined repeatedly with itself. Second, the symmetries of the first, third and fourth shapes all commute with each other, that is, the effect of the combined transformation is the same no matter which isometry is performed first, as opposed to the symmetries of the second that do not. Third, the numbers of isometries that produce the identity transformation when they are combined once with themselves are different for each shape. There are 7 such isometries in the fourth shape, 5 in the second, 2 in the third and only 1 in the first. A representation of the symmetry elements of the four shapes is shown in figure 2. The symmetry elements of the four shapes can be pictorially represented as points, lines and planes, that is, as the geometric loci of points that remain invariant under the repeated application of a specific symmetry transformation.

The information about the symmetries and the ways they are combined with each other is captured in the definition of the various symmetry groups. The mathematical study of the groups has been given in many sources (for a recent account, see for example, Armstrong, 1988). The relation of groups to design and art in general has been treated in several sources (for classic accounts in the field see, Weyl 1952, Shubnikov and Koptsik 1972, March and Steadman 1974).

Briefly, a group is a set endowed with a rule; the set can be any collection and the elements of the set are whatever comprises this collection. The rule combines any ordered pair x, y of elements of the set and obtains a unique product xy which also lies in the set; from this definition it follows that both possible ways of combining any two elements, x, y, that is, xy and yx, also lie in the group. The rule is usually referred to as a multiplication or a composition on the given set. The structure of a group is the statement of the results of all possible compositions of pairs of elements.

A group is a set G together with a rule on G that satisfies three axioms: (a) The multiplication is associative, that is to say, $(xy)z = x(yz)$ for any three, not necessarily distinct elements in G. (b) There is an element e in G, called an identity element, such that $xe = x = ex$ for every x i G. (c) Each element in G has an inverse x^{-1} which belongs to the set G and satisfies $x^{-1} x = e = x x^{-1}$.

In general $xy \neq yx$; certain pairs of elements x, y in G obey $xy = yx$ and is said that these elements commute. The identity element e commutes with all elements of a group and every element commutes with its inverse. If all elements in G commute with each other, i.e., $xy = yx$ for all x, y of G, the group G is called commutative or Abelian.

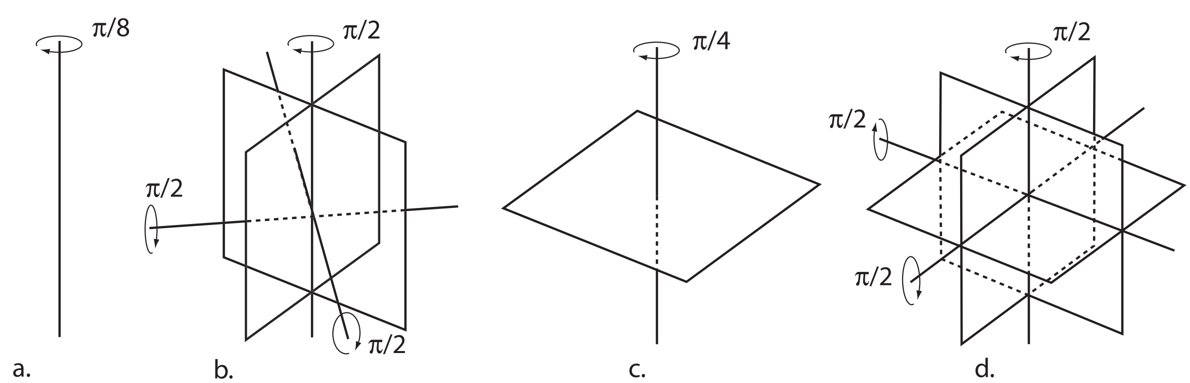


Figure 2. Pictorial representations of the symmetry elements of the four shapes of figure 1. (a) Shaved octagonal truncated pyramid (b) Isosceles tetrahedron (c) Shaved square prism (d) Rectangular prism.

A group is abstract if its elements are abstract, i.e., if they are not defined in any concrete way. A concrete example of an abstract group, i.e. a group with concrete elements with a law of composition, is called a realization of that abstract group. Such realizations might be groups of numbers, matrices, or geometric transformations. There is a finite number of abstract groups of a given order n (Budden, 1972); for example there are only 5 abstract groups of order 8 but only 4 of those have realizations that pertain to symmetry considerations. These 4 are the ones that have been used so far in this paper to illustrate key concepts and tools of group theory. These 4 groups are instances of the 4 infinite types of abstract groups that provide the structures of all finite shapes in three-dimensional Euclidean space, except the platonic solids and their derivatives. These 4 abstract groups are the abstract cyclic group C_n of order n, the abstract dihedral group D_n of order 2n, and their direct products with the abstract cyclic group C_2, that is, the group $C_n x C_2$ of order 2n and the group $D_n x C_2$ of order 4n. These 4 infinite types of groups together with the 6 more finite abstract groups that capture the symmetries of the platonic solids and their derivatives provide 10 algebraic structures that capture the symmetries of any finite shape in any Euclidean space up to dimension 3 (Armstrong 1988).

The correlation of the abstract groups with their realizations in Euclidean space is not an easy task. The greatest problem that arises is that there is no clear and intuitive mapping between the abstract groups and the symmetry groups in three-dimensional space. In two-dimensional space there is no such problem and indeed the abstract groups and the symmetry groups coincide. For example, when Weyl asserted that Da Vinci systematically explored dihedral symmetries in his designs for central churches it is clear that the designs are conditioned by a dihedral group structure and exhibit reflectional and rotational symmetries. In three-dimensional space no assertion of this kind can be made. If a three-dimensional design exhibits a dihedral group structure, it could be that it may have reflectional and rotational symmetries, or it may have only rotational symmetries, or it may have additionally rotor reflectional symmetries.

In three-dimensional space the 10 algebraic structures that capture the symmetries of any finite shape are realized by 14 types of symmetry groups (Yale 1964). These 14 types of symmetry groups split into 2 types: 7 infinite types and 7 finite types. The 4 infinite abstract groups capture the symmetries of 7 infinite classes of symmetry, the so-called prismatic groups that describe the symmetries of any solid that does not have a greater than a 2-fold rotation axis perpendicular to their major axis. The 6 finite abstract groups capture the symmetries of 7 finite classes of symmetry, the so-called polyhedral groups that describe the symmetries of the platonic solids and their variations.

A pictorial representation of all symmetry classes up to the order of 8 of the 7 prismatic three-dimensional symmetry groups is shown in figure 3. The 7 infinite prismatic groups split into 4 types, namely, 2 abstract cyclic groups C_n, 3 abstract dihedral groups D_n, 1 abstract group composed by the direct product of a cyclic group of order n and a cyclic group of order 2, $C_n x C_2$, and 1 abstract group composed by the direct product of a dihedral group of order n and a cyclic group of order 2, $D_n x C_2$. The four shapes discussed so far all exemplify a unique type of each one of these 4 algebraic structures. A pictorial representation of the structure of the shaved octagonal truncated pyramid is shown in figure 3(a), of the isosceles tetrahedron in figure 3(c), of the shaved square prism in figure 3(f), and of rectangular prism in figure 3(g). In this figure and all subsequent figures simple labels in the form of an open or closed circle are associated with the faces of the shapes to break down the symmetry of the shapes. Labels associated with the front face of a shape are closed and labels associated with the back face of a shape are open.

3 Nested structures

A lot of the interest in symmetry groups and their corresponding designs lies in the nested structures observed in the group structures and their corresponding designs. For any group consisting of n elements there are n^2 possible subsets that some of them satisfy the group axioms and others

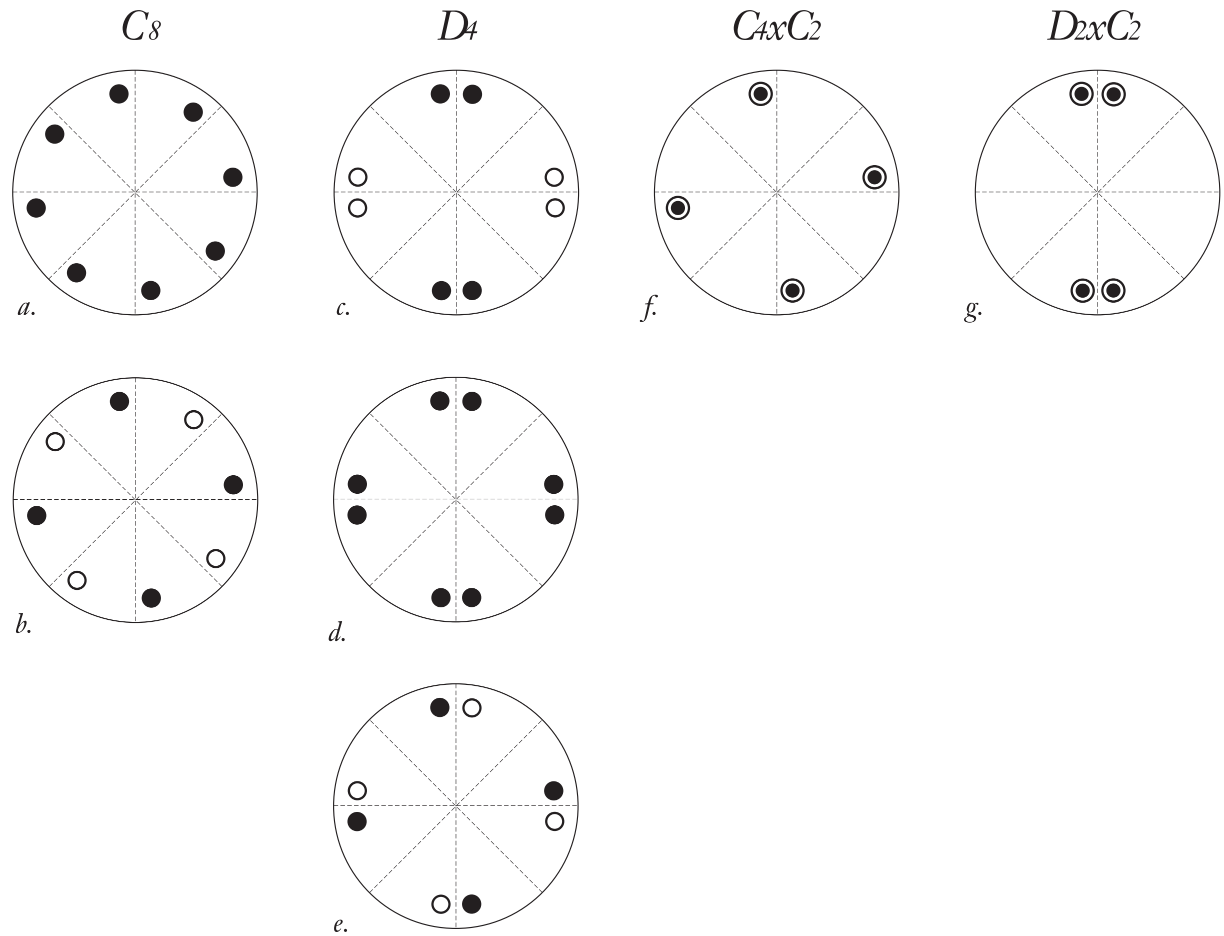


Figure 3. Diagrammatic representations of all symmetry classes of order 8.

not. The best tool for a first scanning of possible symmetry subgroups is given by the Lagrange's theorem that states that if H is a subgroup of a group G, and if the order of G is n, then the order m of H is a factor of n. In other words, the theorem specifies that the order of a finite group is a multiple of the order of any subgroup. From this follows that all prime-order groups have no proper subgroups. Conversely, any two finite groups can be related through a supergroup. The relation of the 4 groups that capture the symmetries of the four shapes discussed so far is shown in figure 4.

There are several techniques that are used in the identification and generation of subgroups. A specific type of subgroups can be immediately picked out because every element of a group may be used to generate a cyclic subgroup. Given a group G and an element x of G, the set of all powers of x is a subgroup of G. This subgroup is called the subgroup generated by x and is written as <x>. If x has finite order m, then $<x> = \{e, x^1, x^2, x^3, \dots x^{m-1}\}$. If x has infinite order, then <x> consists of infinite elements. In both cases the order of x is precisely the order of the subgroup generated by x. If there is an element x in G such that <x>=G, then G is a cyclic group. Similarly, any subset of a group may be used to generate a subgroup. Given a group G and two elements x, y in a subset H of G, the set of all powers of x and y and their combinations is a subgroup of G. An expression of the form $x^m y^n$ for m, n any integers is a word in the elements of H (Grossman and Magnus 1964). The collection of all these words is a subgroup of G. This subgroup is called the subgroup generated by H and written <H>. If there are elements x, y in H such that <H> = G, then the set H is a

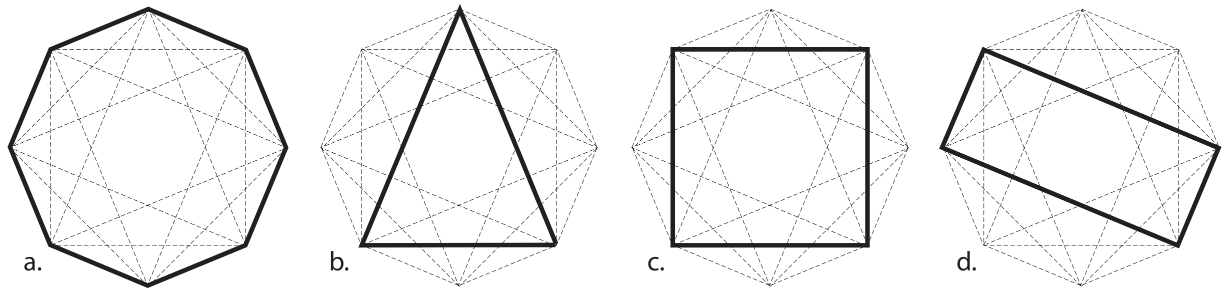


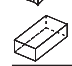
Figure 4. The underlying group structure of all four shapes of figure 1. (a) Shaved octagonal truncated pyramid (b) Isosceles tetrahedron (c) Shaved square prism (d) Rectangular prism.

set for generators for G. The idea of a group generated by one or two elements may be extended for any number of generators.

The concept of nested subgroups is used to illustrate the four symmetry structures discussed so far. All symmetry groups have 8 elements each. The total number of their subsets is 2^8= 512 subsets. Among those subsets few qualify to be subgroups of these four groups. Table 1 provides the number of all subgroups for each type.

The set of the symmetry subgroups of a particular symmetry group can be further sorted by a relation that orders all the subgroups in the set. If this relation can be established for all pairs of elements in the set then this relation is called total or strict order and the set is called chain. For instance, the relation "less than or equal to" (=) is a total order on integers, that is, for any two integers, one of them is less than or equal to the other. If this relation is defined for some, but not necessarily all, pairs of items, then the order is called partial order and the set is called a partial ordered set or poset. For instance, the sets {a, b} and {a, c, d} are subsets of {a, b, c, d}, but neither is a subset of the other. In other words, the relation "subset" is a partial order on sets. Formally, both total order and partial order are relations that are reflexive, transitive and antisymmetric: Reflexive is a

Table 1. Subgroups of all symmetry classes of order 8.

Shape	Name	Symmetry Class	Subgroups
	Shaved Octagonal Pyramid	C_8	4
	Isosceles Tetrahedron	D_4	10
	Shaved Square Prism	C_4xC_2	8
	Rectangular Prism	D_2XC_2	16

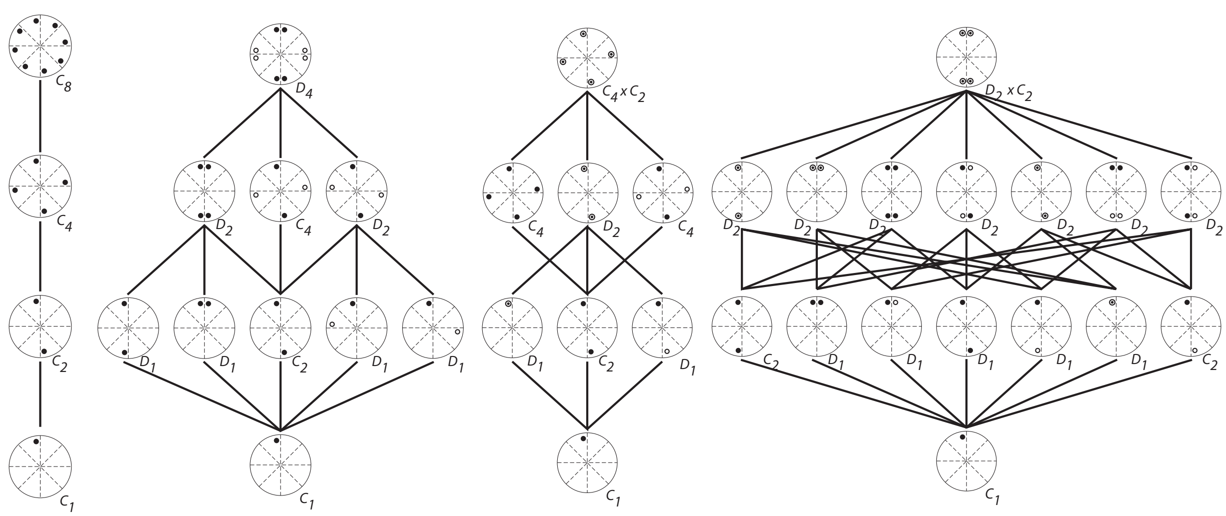


Figure 5. Partial order of all symmetry subgroups of the four shapes in figure 1. (a) C_8 : Shaved octagonal truncated pyramid (b) D_4 : Isosceles tetrahedron (c) C_4xC_2 : Shaved octagonal truncated pyramid. (d) D_2xC_2 : Rectangular prism.

binary relation R for which a R a for all a. Transitive is that binary relation R for which a R b and b R c implies a R c. Antisymmetric is a binary relation R for which a R b and b R a implies a = b. Typically graphs are used to represent such order and show the nested relations of the subgroups diagrammatically in lattice diagrams. In graph representation, an empty relation between elements is represented by a graph with vertices and no lines connecting them and a complete relation between elements is represented by a graph with vertices that are all connected one to another. The graphs or lattices of the chain of C_8 and the posets of D_4, C_4xC_2 and D_2xC_2 are shown in Figure 5.

4 Applications

Symmetry groups and their nested subgroups provide an excellent framework for systematic studies in the analysis and synthesis of form. In both types of studies symmetry groups may be used to provide the nested underlying structures for the constructive description of a design. The researcher or the designer may select a specific schema or schemata that contain specific isometries of a given symmetry groups to either analyze existing designs or construct new designs that are built upon the elements of the nested symmetry groups. The power of this methodology that relies on partial order sets of parts of designs for analysis or for creative composition is that it brings forward specific relationships between parts and to the whole that are of interest to the researcher or the designer without paying attention to the overall symmetry of a configuration. The overall design may have or may not have specific symmetries; it is the ordering of the parts that counts and the relationships between them.

This approach departs significantly from established paradigms in analysis and synthesis in visual arts and the arts in general whereas symmetry typically is associated with overt repetition and immediate recognition of the total formal structure. Examples of such appropriations of symmetry in analytical and creative studies include the explorations of configurational possibilities of central-plan churches by Da Vinci (Richter 1970), as well as Sullivan's ornamental designs built upon symmetries of basic geometric forms (Sullivan 1924). Still other modes of formal composition in the arts require that symmetry will be followed employing more esoteric isometric transformations rather than typical reflections and translations, that is, isometries such as glide reflections, rotor reflections, screw rotations, or even rotations without the presence of any reflections. Typical examples of such appropriations of symmetry in creative composition are found predominately in the twentieth century architectural design and primarily early modernism in the works of Le Corbusier, F.L Wright and their followers (March and Steadman, 1974). Still these latter design examples while they definitely extend the discourse employing consistently new transformations in composition, still they are subordinate to the very same ideologies that were previously associated with symmetry, and the outcome at the end is still a total reliance of the whole design upon a given structure.

The approach outlined in this paper departs from other approaches in the field in two significant ways. First it bypasses the routine understanding of symmetry as overt repetition and apparent structure and all relevant ideologies tied up with these notions. Such understanding of symmetry is obviously within the discourse but it does not comprise the whole horizon of inquiry. Second it lends itself so much in analysis or in design without specifying in advance that the final product should be symmetrical. The model is extremely appropriate in analytical studies whereas designs evoke a sense of order but at the same time remain elusive as to their underlying structure or structures. Excellent recent examples in analysis include a series of analytical studies of designs by Palladio and Alberti (March 1998). A nice overview of this model for analysis of two-dimensional representations of designs has been given recently by Park (2000).

A constructive program for the employment of this model for the generation of designs based on given symmetry groups is briefly outlined below and its main differences with similar established methods are sketched as well. The program is built upon the original constructive program for the definition of languages of design using spatial algorithms (Stiny, 1980). The program has six stages: (1) A vocabulary of diagrams. (2) A vocabulary of shapes. (3) A set of spatial relations specified between the shapes and the diagrams in the vocabulary. (4) A set of shape rules specified in terms of the spatial relations and their correlation to the diagrams. (5) An initial shape comprised by some shapes in the vocabulary. (6) A set of shape grammars specified in terms of the shape rules and the initial shapes. The individual stages of the program are shown in figure 6.

The novelty of the program lies upon the interaction of sets of diagrams of symmetry groups and sets of spatial relations. Spatial relations bring forward the spatial ideas that are employed by the designer of the grammar. Diagrams of partial ordered sets of symmetry groups bring forward the conceptual notions of layering and superimposition of various parts of designs. It is the interaction of these two types of shapes, spatial relations comprised by points, lines, planes and solids, and

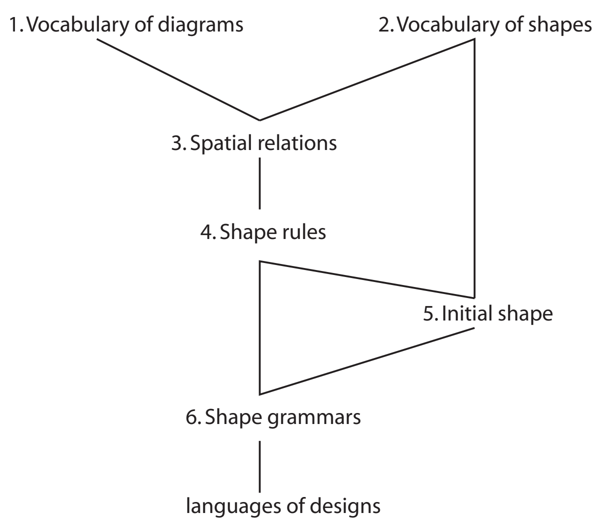


Figure 6. A constructive program for the definition of languages of designs based on diagrams of lattices of symmetry groups.

diagrams comprised equally by points, lines, planes and solids, that lead the computation for the generation of languages of designs. In the first case spatial relations specify the spatial order of the design; in the latter case diagrams specify the transformational order of the design; in the first case the parts are freely chosen or constructed by the designer; in the latter case the parts are selected from a predefined vocabulary of lattices of symmetry groups.

A brief study illustrating the concepts mentioned so far is shown below. The shapes selected for this exercise are taken from the Froebel building gifts 3-6 as a tribute to the ongoing research on formal composition using these shapes as powerful vehicles for the illustration of the constructive power of shape grammars in the generation of form (Stiny 1981, Knight 1992, March 1995, Economou 1999). Still in this case they are all parameterized to allow for any proportions between their faces. The diagram selected for the derivation of this simple spatial study is the lattice of D_4. The partial order of the symmetry group D_4 is chosen to illustrate an alternative use of the structure of the square in three-dimensional space. In this series of spatial studies one spatial relation would suffice to instantiate all subgroups of the lattice as long as the rule applied under scale transformation; here however all spatial relations are different for the various subgroups of the lattice of the D_4 to make explicit the spatial differences of the various parts of the design and facilitate the reading of the lattice. A set of spatial relations used in this example is shown in figure 7.

Shape rules are defined in terms of spatial relations between shapes in the vocabulary and between diagrams in the vocabulary. The former

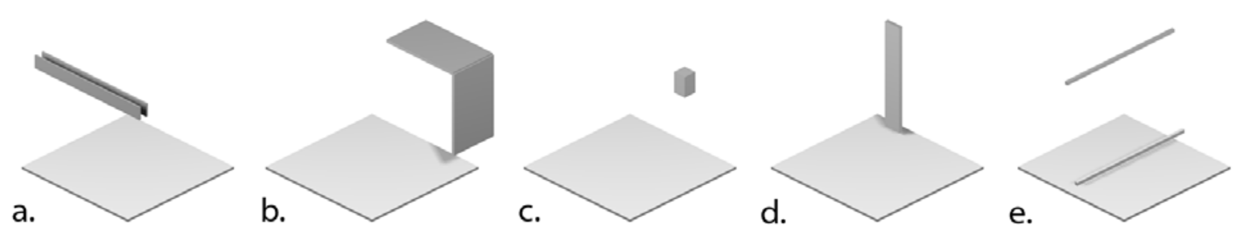


Figure 7. A set of spatial relations between shapes in the vocabulary.

rules control the spatial characteristics of the design. The latter rules control the order of the design in terms of its various parts. Both types of shape rules along with the initial shape of the lattice of D_4 comprise a shape grammar and the set of designs generated by the recursive applications of the rules in the grammar forms a language of designs. Figure 8 shows four views of a design in the language using the devices mentioned so far. The decomposition of this design in the language in terms of its underlying D_4 group structure is shown in Figure 9.

5 Conclusions

The notion of nested symmetries whereas seemingly complex designs can be understood as superimposed layers of simpler designs was a major focus of this paper. To this extent extensive illustrations of specific symmetrical structures in three-dimensional space were presented as tools for further studies in the analysis and synthesis of three-dimensional form. There is no doubt that the orthogonal projections of plans, sections and elevations are a magnificent source for such a formal analysis. It is hoped that an extension of the same formal techniques for creative composition in three-dimensional design space is readily available.

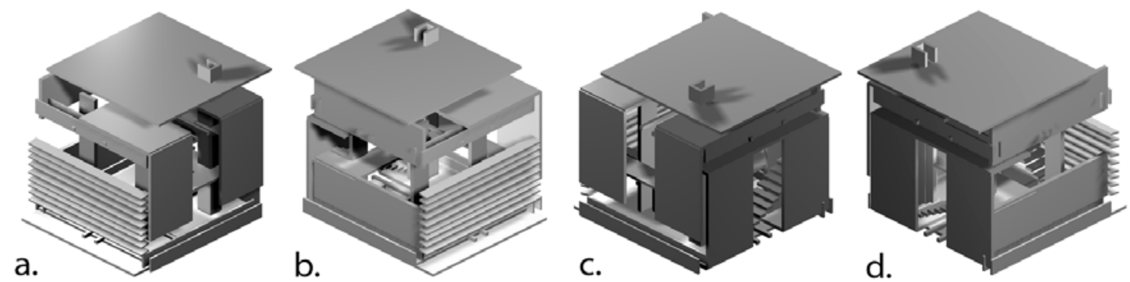


Figure 8. Axonometric views of a design in the language

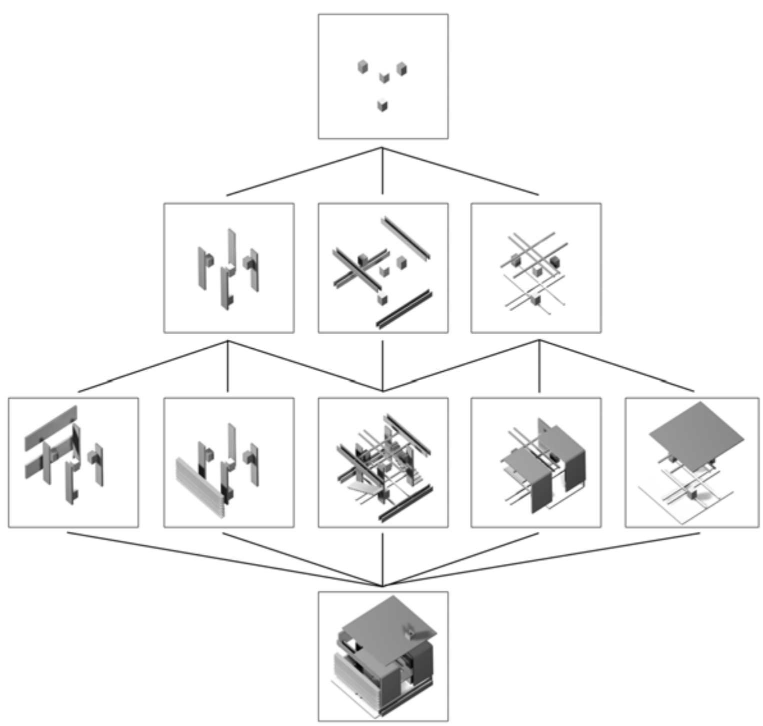


Figure 9. Partial order set of parts used for the generation of the design in figure 8.

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