# Quasi-Projection: Aperiodic Concrete Formwork for Perceived Surface Complexity

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## ABSTRACT
Aperiodic tiling patterns result in endlessly varied local configurations of a limited set of basic polygons, and as such may be used to economically produce non-repeating, complex forms from a minimal set of modular elements. Several well-known tilings, such as by Penrose (2D) and Danzer (3D), have been used in architecture, but these are only two examples of an infinite set of possible tilings that can be generated by the projection in two or three dimensions of high-dimensional grids subject to rotations. This paper proposes an interface that enables the user to parametrically search for such tilings. Assembly rules are explained by which arbitrary geometry as specified by NURBS surfaces may be based on the pattern to form a non-repeating complex surface. As an example, the fabrication in concrete of a cylindrical tiling is used to demonstrate the mass production of a continuous, free-flowing structure with the aid of a minimum amount of formwork.

Keywords: Quasicrystals, aperiodic tiling, strip projection method, assembly rules, tangential continuity, formwork, modularity.

## 1 INTRODUCTION
In The Self-Made Tapestry, Philip Ball defines pattern as “arrays of units that are similar but not necessarily identical, and which repeat but not necessarily regularly or with a well-defined symmetry” (Ball 1982). Patterns are rendered by external forces and travel extensively in space, motifs are static and self-contained. Quasi-periodic structures were discovered in 1982 within the field of crystallography from electrons diffraction disclosing patterns with icosahedral symmetry (DiVincenzo and Steinhardt 1999). They are special crystals with no translational symmetry. That is, contrary to crystals they cannot repetitively align themselves as tiles or building blocks to fill up space without resorting to rotational symmetry. Crystals have close-range order, whereas quasicrystalline structures disclose long-range order even though they are comprised of only few different polygons. A 2D Penrose tiling portrays a similar structure to those found in Quasicrystals. Growing interests in aperiodic structures have been applied in recent years within the field of design. The Water Cube (National Swimming Centre) for the Beijing Olympics by PTW Architects and Arup, the RMIT Storey Hall in Melbourne, Australia, by ARM Architects, or Aranda/Lash’s quasi-furniture (Aranda and Lash 2006) are some examples. Philip Beesley’s Orgone Reef installation (Beesley 2005) uses a Penrose tessellation as a subtle underlying geometry to efficiently organize its intricate hybrid fabric and allow its structure apparent freedom from periodic symmetry. Architecture profits from the intrinsic properties of aperiodic formation as it provides modularity without the repetition of the same module throughout (e.g., Erwin Hauer architectural screen [Hauer 2007] or a singular object built up of many variant components (e.g., Greg Lynn BlobWall [Lynn 2008]. In all cases, the patterns are used to generate apparent variety of form from a limited set of basic polygons. This they do economically and precisely. The well-known tilings by Penrose (Penrose 1989) and Danzer (Danzer 1989) are only two examples of an infinite set of possible tilings. In 1981, N.G. De Bruin (De Bruin 1981) showed that the Penrose pattern could be seen as the projection of an object in 5-dimensional space onto a plane, and Marjorie Senechal (Senechal 1995) suggested an early implementation, asking Eugenio Durand to develop a program to generate Penrose tilings using the projection method (Durand 1994). Related methods like the Updown generation method (Austin 2005) and the dual grid or Pentagrid method (Duffy 2004) for generating such tilings also exist. This paper proposes an interface that enables the user to parametrically search through this infinite set of such tilings, and thus generate a base on which to form a non-repeating complex surface that can be used in architecture.

## 2 THE STRIP PROJECTION METHOD

The basis of the projection method consists of selecting from a regular n-dimensional grid all the points falling within an area defined by a clipping boundary (the strip), and then of projecting the selected points onto a lower dimensional sub-space.

### 2.1 FROM 2D TO 1D (X, Y)

In figure 1b, a set of points are projected from a 2-dimensional grid onto a 1-dimensional line. If the gray segments were labelled A and the dark ones B, the angle between the line and the grid axis results in a series of this sort: {…ABABAABABAABABABAAB…}. Although only the two units A and B are used throughout, their arrangement does not form a group that repeats itself across the series. This in contrast to strings disclosing clear translational symmetry such as {...BBBBBBBBBBB…} in figure 1a or {...AABAABAABAAB…}. The latter can be produced by setting the clipping boundary in perfect alignment with an axis, or at an angle exactly through a given series of points, but any rotation defined by irrational numbers, as they are not expressible as whole number fractions, will result in an aperiodic pattern. The number of these angles is infinite, and changing from one to the next allows a different pattern to emerge. Just as this single rotation angle is the controlling parameter for the one dimensional set, the parameters of rotation in higher dimensions will completely specify two or three-dimensional patterns. Selection of the points to project is determined by the depth of the clipping boundary. The strip selected is set just wide enough to incorporate only the points falling within one unit of the 2-dimensional lattice. As the highlighted strips show in figure 1a and 1b, it is not a constant amount, but must vary when the lattice is rotated due to the changing depth of the unit diagonal. The strip is defined by two boundary lines:

Y1= 0; fixed at the origin

Y2= [min. to max.]; shifting from 1 to sqrt(2) upon x-y rotation

In solving for this, a depth vector was created on the diagonal of a single grid unit (from (0, 0) to (1, 1)) and multiplied by the rotation matrix. The second boundary line is then defined as the y-value of this depth vector, and all points of the lattice smaller or equal to this are used in the projection.

### 2.2 FROM 3D TO 2D (X, Y, Z)

Exactly the same method is applied to perform the projection from the third dimension, where there are three possible rotations: in the X-Y, Y-Z, and Z-X plane. For each rotation exists a specific rotation matrix, which when multiplied by a vector, will cause that vector to rotate. In practice, the three rotations are multiplied into one overall rotational matrix. Below is an example of a 3D rotation matrix around the Y-axis or the X-Z plane, where Alpha can parametrically range from 0 to 360 degrees.

$$(x,\ y,\ z,\ 1) * \begin{array}{c|cccc|} & x & y & z & a \\ x & cosa & 0 & -sina & 0 \\ y & 0 & 1 & 0 & 0 \\ z & sina & 0 & cosa & 0 \\ a & 0 & 0 & 0 & 1 \end{array} = \begin{array}{cccc} x & y & z & a \\ (x*cosa+ z*sina, & y, & -x*sina+ z*cosa, & 1) \end{array}$$

The same principle also applies for higher dimensions. In 4 dimensions (x, y, z, a, b), there are 6 possible rotation matrices (M 5, 5) in each plane: x-y, x-z, y-z, x-a, y-a, z-a. In 5 dimensions (x, y, z, a, b, c), there are 10 possible rotation matrices (M 6, 6) in each plane: x-y, x-z, y-z, x-a, y-a, z-a, x-b, y-b, z-b, a-b. In 6 dimensions, there are 15 possible rotations; in 7 dimensions, 21 possible rotations, etc. The values for each of these rotations define the resulting pattern uniquely and precisely, and when treated as parameters, they allow the generation of any of the infinite set of aperiodic patterns.

## 3 ENSURING THE DEPTH OF THE CLIPPING BOUNDARY

The depth vector for a 3D to 2D projection is from (0, 0, 0) to (1, 1, 1). The width of the strip is meant to be the total depth of a single unit of the grid, but the first quadrant unit diagonal (1, 1, 1) only functions as an appropriate clipping depth when rotation is within the first quadrant. Beyond this, the diagonal will have rotated too far and may have an overall depth less than another diagonal vector (e.g. (1, 1, -1)) or may even point behind the notional hyperplane of projection (a hyperplane is an n-dimensional generalization of a plane; an affine subspace of dimension n-1 that splits an n-dimensional space (wiktionary.org / accessed June, 2008)).

What needs to be avoided is a negative depth vector. A pre-emptive measure to this problem is to verify the signs of its coordinates before rotations. This was done by checking against its z-value (0, 0, 1), since clipping occurs along the z-axis (fig.1c, 1d), and by multiplying it by the inverse of the overall rotation matrix (the inverse of a rotation matrix is its transpose; flip along its diagonal). If any of its coordinates turned out negative (e.g. (0.651, -0.781, -0.265)), then the sign(s) of the corresponding coordinates of the original depth vector are flipped accordingly (1, -1, -1). As shown in figure 1e, boundary errors can become significant in preventing a tiling from being entirely projected and from continuously progressing under any rotational changes. There, Z2 was 1.09 units and its depth_Vec was in the SW quadrant whereas in figure 1f, the adjusted Z2 is now 1.58 units and its depth_Vec in the NE quadrant.

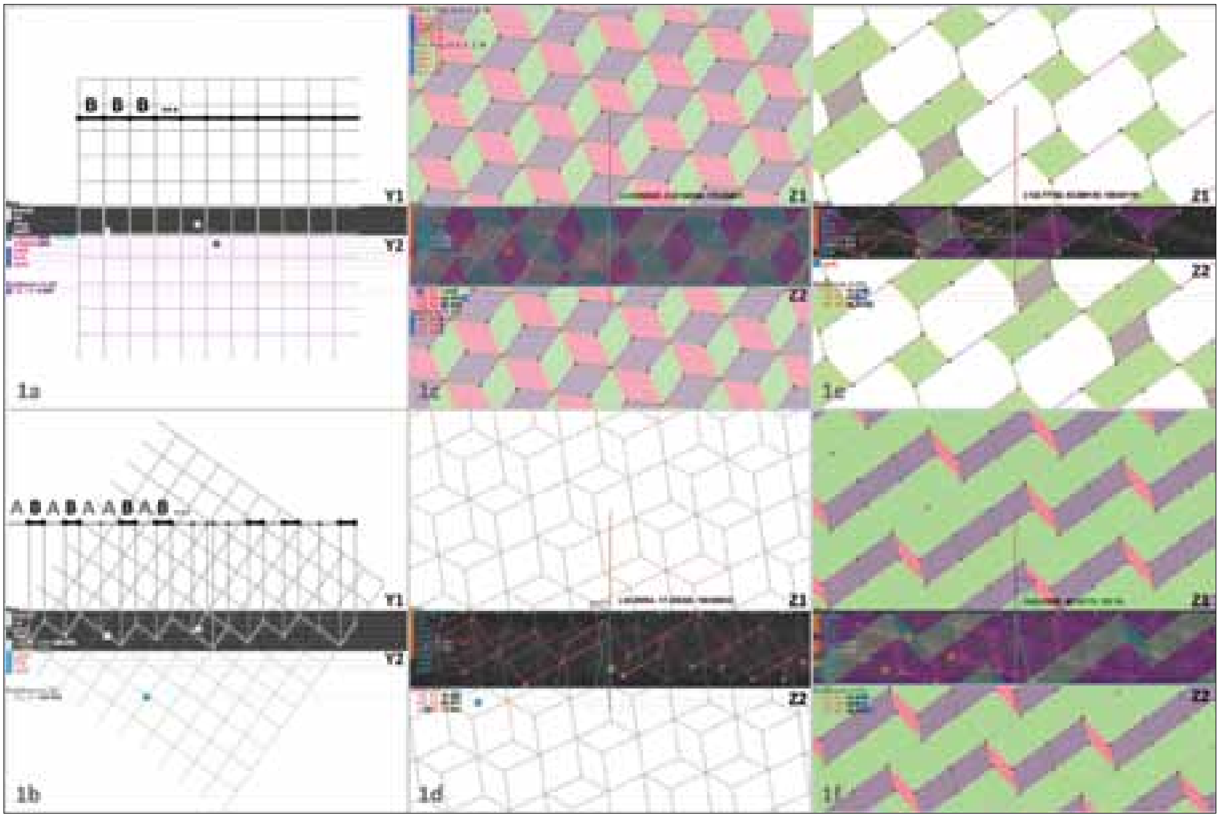


**Figure 1: 1a** 2-dimensional lattice projected in 1 dimension; **b** Same as **1a** but after rotation disclosing an aperiodic structure; **1c** Periodic tiling under specific rotation. The highlighted strip is the X-Z projection of a **3d** lattice showing the space between the two clipping planes (z1, z2) for points selection; **1d** Aperiodic tiling from a **3d** grid projected in **2d; 1e 2d** projection on the x-y plane of a **3d** grid with incomplete tiling; **1f** Same as 1e but with the corrected depth vector making a complete tiling.



## 4 CLIPPING MORE THAN ONE DIMENSION

Existing literature (Durand 1994, Weber 1997) on the subject presents the projection method with the aid of a 2d to 1d projection diagram (as in fig.1b), but is somewhat ambiguous about the description of the clipping offset for higher dimensions. The problem can be visualized in the case of projection onto a line. When clipping occurs from 2d to 1d it is within an area clearly defined by the line of projection and another offset from it. If one attempts to project from a 3d grid to a 1d line however, there are several possible definitions for this offset, two of which can potentially be used to determine the clipping boundary. One will be referred to here as the rectangular method, the other as the cylindrical method. The rectangular approach is that the line of projection is actually identified as the intersection of two planes (or hyperplanes) defined by the dimensions that are to be clipped, in which case the offset is given by offsetting each of these planes. The clipping boundary is that rectangular area bounded by the four planes. The second definition takes the line of projection as just a simple line and its offset is determined by all the points that are equidistant to it in three dimensions, i.e. a cylinder.

### 4.1 FROM 4D TO 2D (X, Y, Z, A)

In 4-dimensional space, with 6 possible rotations, two dimensions need to be clipped against (z, a) in order to be projected in 2d (x, y). The key issue is to establish if the clipping boundaries in those two dimensions are determined individually by the rectangular method or in a pre-combined manner by the cylindrical. The z-a reference plane in the following diagrams (fig. 2a to 2c) helps to visualize the differences between the two approaches. The cylindrical method has a larger selection area upon rotations contrary to the rectangular method. For instance, as the depth vector in the rectangular method approaches the a-axis or the z-axis, the area is progressively being reduced to a line (fig. 2a, 2d). This will inevitably create holes till no points at all will be found within the clipping boundary. The first method was computed by initializing the 16 points (x, y, z, a) making up a one unit 4d hypercube and by updating the clipping planes positions as the 4d points were rotated. Two vectors (z_depth, a_depth) constantly spanned 2 of those 16 points in making sure they always incorporated the smallest z_min and a_min and the largest z_max and a_max (fig 2f).

Point selection for the rectangular method:

If (z1 plane < X.z <= z2 plane) And if (a1 plane < X.a <= a2 plane)
Then X is inside the clipping boundary and selected.

Point selection for the cylindrical method:

If (z1 plane < (sq(X.z) + sq(X.a))*0.25) < z2 plane
Then X is inside the clipping boundary and selected.

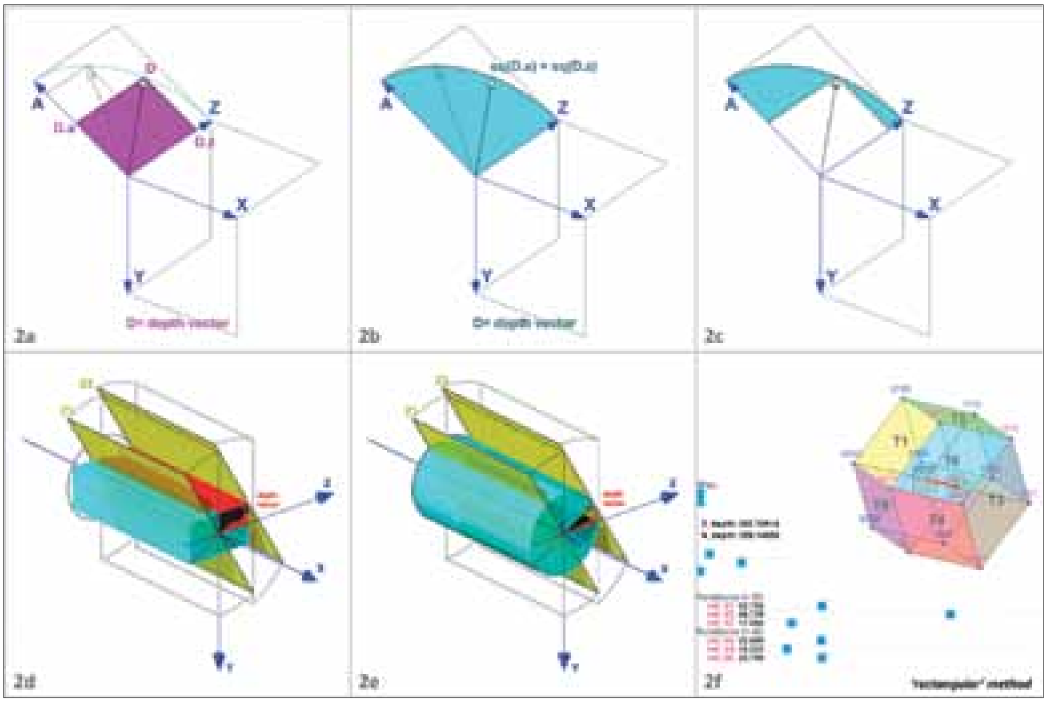


Figure 2: **2a** Diagram of the rectangular method; **2b** Diagram of the cylindrical method; **2c** Area difference between both methods; **2d** Diagram of the rectangular method to define the clipping boundary for a 3d to **1d** projection (line on the x-axis); 2e Same as in fig. **2d** but for the cylindrical method; **2f** 4d hypercube and its clipping plane positions subject to a set of rotations. The red line is the depth vector.

To establish which was best suited, a comparative test was carried out between the two methods. A hypercubic grid of points in four dimensions was directly projected onto a two dimensional plane using each method separately and jumping down two dimensions at once. For the rectangular method, points were missing in 2d upon arbitrary rotations (fig.3a) whereas the cylindrical method always presented the same complete projected tiling (fig. 3b). These showed that only the cylindrical method makes a complete tiling. It was further established (fig. 3c to 3f) that contrary to the rectangular method, the cylindrical performed as well in jumping down two or more dimensions from higher dimensional grids: 5d to 3d (n-2), 5d to 2d (n-3), 6d to 2d (n-4)

## 4.2 DEPTH OF THE CLIPPING BOUNDARY

The depth vector for a 4d to 2d projection goes from (0, 0, 0, 0) to (1, 1, 1, 1). Similarly to the principle used from 3d to 2d, the correct clipping boundary requires positive values in each of the dimensions being clipped against (z, a). A vector of one unit for these dimensions (0, 0, 1, 1) is multiplied by the inverse of the overall rotation matrix to check which quadrant it lies in. If any of its coordinates has a negative sign (e.g. (0.631, -0.579, -0.067, 0.345)), then the corresponding coordinates of the initial diagonal unit vector are consequently flipped (e.g. (1, -1, -1, 1)) to guarantee that when rotated by the overall rotation matrix, the clipping dimensions retain each a positive value.

## 5 ADJACENCY ANALYSIS FOR ADAPTED NURBS GEOMETRY

Found tilings can further systematize the adaptation of other types of geometries without compromising the efficient repetition and distribution of few elements into a diverse field. NURBS tiles are here chosen among other possible geometries to test if while constrained to their initial polygons (e.g., 6 tiles in 4d) and to their inherent assembly rules,



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tangential continuity between tiles can flow free off the tiles' edges into a continuous assemblage of a long-range order. A search through a tiling first determines for each tile's edge which other adjacent edges from other tiles it has in common and allows to group common edges into a series of independent NURBS linetypes. For instance, in figure 4a a minimum of four different linetypes were disclosed from a 4d tiling made of 6 tiles (5 in 5d, 6 in 6d). Although the surface of a NURBS tile is defined by its four NURBS edges, it is made of two sets of different linetypes (fig. 4c). Each line type's control points can be parametrically manipulated in x, y and z (fig. 4d) causing the entire field to readjust upon local changes. Geometrical offsets of two kinds, central to an edge and at the vertices, ensure tangential continuity between NURBS tiles (fig. 4f). In addition, openings were experimented with to engage the two-sidedness of a tiling and its corresponding thickness. Both openings at vertices and at middle of tiles were generated by an offset tiling perforating the top one (fig. 4b, 4e). Figure 4b demonstrates how following these assembly rules, NURBS tiles may extend past a tiling's original polygons and form a non-repeating complex surface. A 5d tiling is composed of ten tiles or less in some cases. The 2d Penrose is such a case. Even though constructed of only 2 different rhombi, it can as well be generated by the 2d projection of a 5d grid where 10 emerging tiles becomes just 2 sets of 5 identical tiles under specific rotations and only calling for one singular linetype. Subject to an adjacency analysis, a 2D Penrose could be thought as a more generic 5d type (10 tiles, 40 edges, 5 linetypes) so that a NURBS geometry can make up for a greater diversity in a field formation from the exact same tiling (fig. 3c).

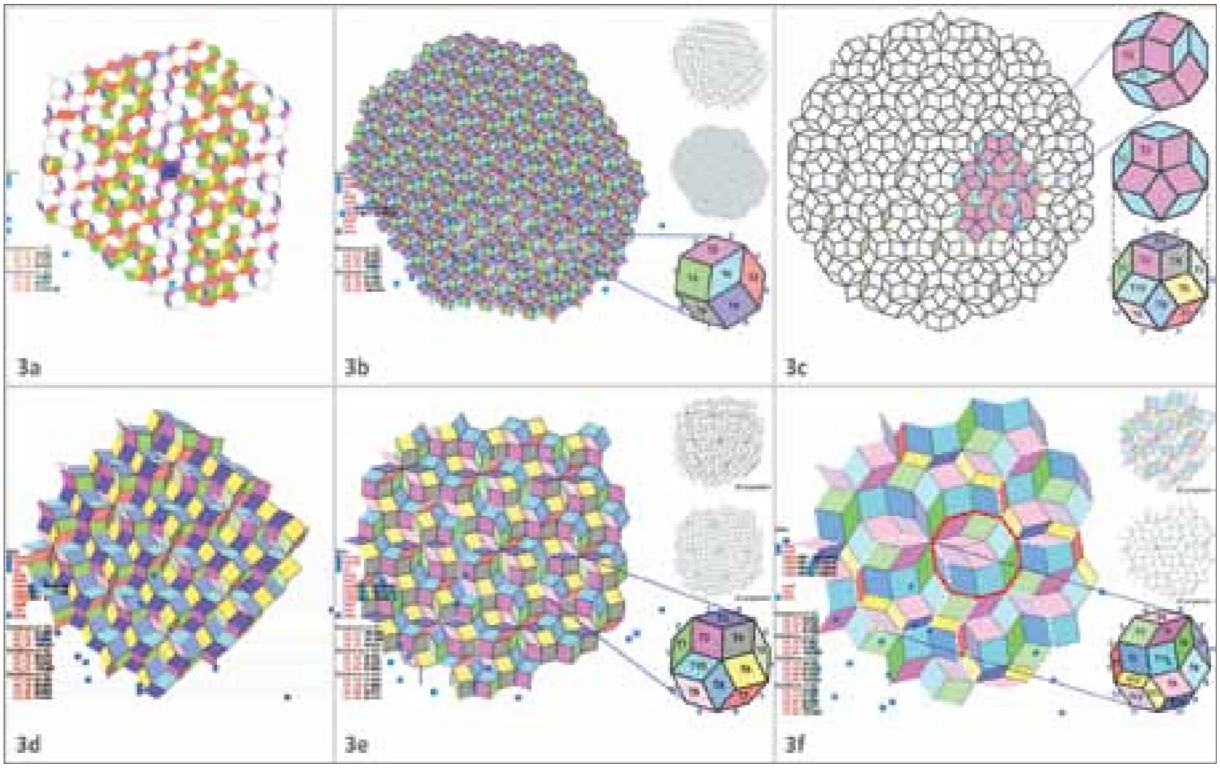


**Figure 3: 3a** Incomplete 2d tiling using the rectangular projection method from a 4d grid (6tiles); **3b** Completed 2d tiling using the circular method; **3c** 2d Penrose from a 5d grid projected in 2d; **3d** Tiling from a 5d grids projected in **3d**; **3e** Tiling from a 5d grids projected in 2d; **3f** Hybrid tiling from a 6d grid projected in 2d (x, y, z, a, b, c)

## 6 A CONCRETE APPLICATION OF A QUASI-CYLINDER

A sub-region of the 2d or 3d projection of an n-dimensional tiling can be rolled into a cylinder and close perfectly on itself without incrementing the number of dissimilar tiles required for a tiling (fig 5a). The length of a quasi-cylinder (both along its rotational axis and transversal to it) for any tiling is limited to lengths for which local periodicity exists in the regions to be joined, and is therefore always finite, but regions of significant length can be found. Any tiles on the cylinder are askew in regards to its central axis (rotated polygons) and describe a surface of double curvature. Taking into account any thickness for a cylindrical tile causes its edges to wrap and twist while perpendicularly converging along the central axis (fig 5c). Figure 5b presents the 6 NURBS tiles' new spatial arrangement (from the geometrical properties of the quasi roller) for the fabrication of a 275cm diameter prototype made of 39 concrete elements(from 6 different tiles). For each tile a polystyrene mould was devised in 8 removable parts (fig. 6d) and reuse for repetitive casting (5x6 and 1x9). The concrete NURBS tiles were then bolted together following tiling's corresponding assembly rules (fig 6c, 6e). Although in this case an edge from a concrete tile can match 5 other edges from different tiles (6x4 edges for 4 different linetypes), tangential continuity is maintained for any local combinations (fig. 6b). Mass production of tiles from the least amount of moulds demonstrated that their cylindrical assembly remained diverse and free-flowing (fig. 6a).

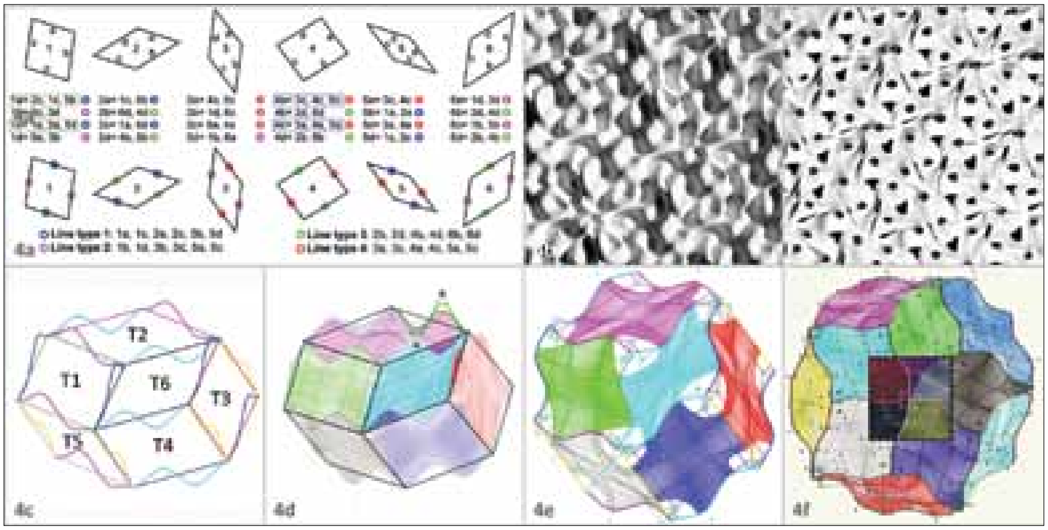


Figure 4: 4a Adjacency analysis from the 4d tiling shown in figure 3b; 4b 5d and 4d NURBS tiling; 4c 6 tiles in 4d showing 4 different line types in sets of 2 for each NURBS tile; 4d Same as 4c with parametric curve line type at edge of tile; 4e.NURBS tiles with openings at the vertices; 4f 5d tiling (10 tiles) with tangential offsets

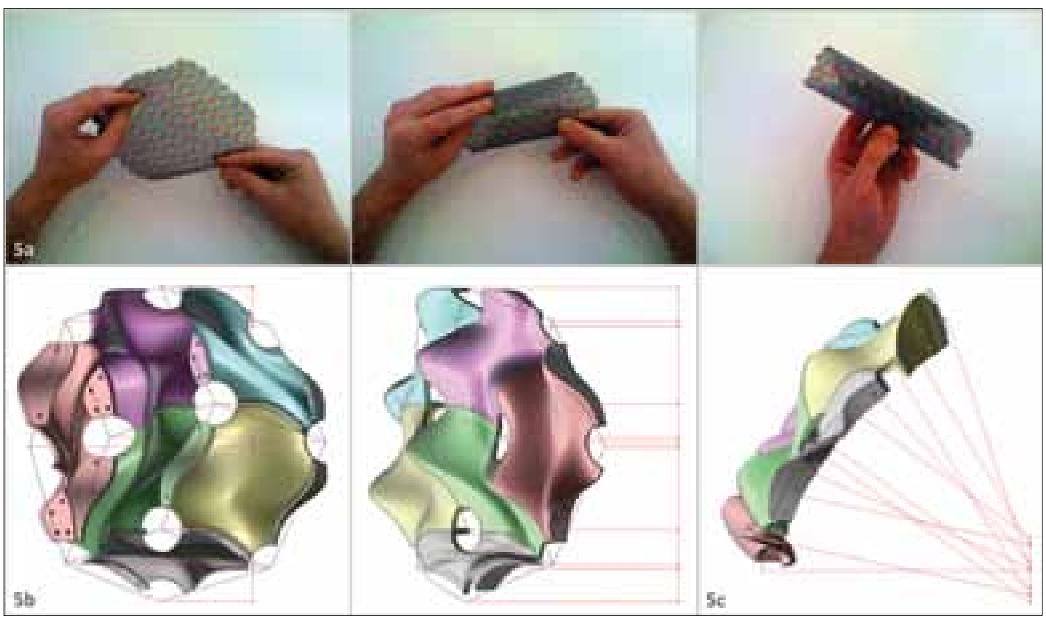


Figure 5: 5a Rolling of a 4d tiling into a cylinder; 5b Views of 6 NURBS tiles with openings; 5c Side view of NURBS tiles converging to the central axis of cylinder

## CONCLUSION

Implications of this as applied in construction are a step toward efficiency of means: the minimum amount of formwork to produce maximum overall complexity. By exposing and probing the space inside which aperiodic structures emerged, the proposed interface extends the quantity of already known tilings to an infinite number of possibilities for any number of higher dimensions. These may be used as scaffolds, on which alternative geometries can smoothly plot their courses past the tiles' boundaries and present relatively complex pattern formations without compromising the scaffolds' efficiency. It has been demonstrated that one can generate modular systems that are not restricted to repetitions of the same periodic pattern, and these were used to enable the mass production of a continuous, free-flowing structure with the aid of few moulds.

## CONTRIBUTORS

ELSA CAETANO FOR THE NURBS PARAMETRIC MODELLING AND THE FABRICATION OF THE QUASI-ROLLER
GUAN LEE FOR THE FABRICATION OF THE QUASI-ROLLER




**Figure 6: 6a** Field model of 4d cylindrical NURBS tiles; **6b** Close up of prototype showing exterior flow between elements; **6c** Close up of interior assembly; **6d** Polystyrene part moulds for concrete casting; **6e** Full scale prototype of quasi-roller

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