SYMMETRY AND EQUALITY OF SHAPES

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Abstract. Shapes are equal if the spatial relations between their parts are equal. But spatial relations are equal if shapes characterizing the relation are equal. This recursive nature of shape and shape relations is examined. A repeated shape has distinct parts each equal and in one of a specified set of spatial relations to another part. A symmetrical shape has the same spatial relation to each part of some repeated shape. Repeated shapes are simple examples of generated shapes but can show intricate behaviour and indicate the problems of predicting spatial properties in generative shapes.

1. Introduction

Symmetry is perfection rarely achieved. Shapes repeated in patterns need to 'add up' in highly constrained ways to get results which display conventional symmetry. A near or partial symmetry of pattern is expressed by selecting parts symmetrically or completing 'gaps' (Economou 1999, Park 2000). We examine repeated shapes and the types of spatial relation created. The spatial relations are regarded as a key element in visual representations rather than coordinate systems. This reflects a commonplace that the 'content' of a visual representation arises from the relations between its parts, rather than absolute positions. The shapes themselves reference and frame one another. In exploring this view we concentrate on shapes themselves rather than their symbolic or diagrammatic surrogates. Reasoning with shapes looks for new relations consequent on given relations. Symmetry arising from repeated parts is one specific and very particular example. In order to unravel what reasoning might mean in this case we take a close look at spatial relations. Further, shapes are changed by rules which are themselves expressions of spatial relations.

Shapes regarded as symmetrical have repeated parts whether molecules in a crystal lattice, spaces in an architectural composition or shape symbols and icons in a diagrammatic representation. Equal parts are repeated in equal ways to construct a symmetric shape. This view looks towards the
The internal structure of a shape is represented in terms of the parts in its description (Stiny 1994, Earl 1997) using spatial relations. Representations of shape are usually in terms of its parts, whether these are the atomistic parts of points and sets of points or the more discrete parts formed as shapes themselves. These latter can be shape elements such as lines and plane segments or more complex aggregates. As parts become more complex there is less likelihood that they will be equal shapes. There are interesting issues here about the complexity of a shape and its representation as parts, but these go beyond the limited scope of this paper. A discussion on computational complexity of diagrammatic representations can be found in Lemon and Pratt (1997). We are going to concentrate on the elementary issues of shape relations, repetitions and symmetry.

This examination of symmetry in shapes first considers how shapes are related, then how the equality of shapes and the equality of relations can be defined. In this way we get inside the structure of parts of a shape, taking a relative view of shapes. The relative disposition of parts describes a shape. The spatial relations of a shape to other shapes defines its 'location'. This might be called an intrinsic view where shapes themselves, their properties and their mutual relations are used to describe other shapes. These intrinsic descriptions recognize another relativism, namely that of the viewer or user to determine the description of the shape. The power of these descriptions is that any intervening apparatus and constraints on representation are minimized whilst the scope for constructing and interpreting them is maximized. The parts in a description are not absolute entities defined and fixed by convention, convention or even mundanely by geometric coordinate systems, but only by their relation to one another and their users or creators. Shape rules provide the constructive mechanisms for generating shapes and creating descriptions by parts (Stiny 1994).

This paper has three main purposes. First, several issues about the definition and representation of shapes and spatial relations are reviewed which bear on the problems of the relative descriptions of shapes. Second, shapes with repeated parts are examined and connections with symmetric shapes established. Third, open questions about the properties of shapes composed of repeated parts are raised. These may be useful in indicating what is special about the relativistic way of describing shapes and why reasoning with shapes is so different from algebraic and arithmetic operations. The questions posed may be trivial and their answers may not be significant in their own right, but they, and questions like them, should help to build a formal picture of how shapes behave. Some apparently
simple theoretical problems in shape description, especially when parts and relations rather than point sets and coordinate systems are used, are not easy to answer. This paper is presented as a discussion offering many questions but few answers.

These problems are relevant to understanding visual representations in so far as they tell us something about the structure of parts of a shape. This structure of relations among the constituent parts of a visual representation is one basis for their meaning. The juxtaposition of elements or parts is an essential mechanism for conveying meaning in visual representations. We do not consider any specific meaning, particular application or type of visual representation here. We limit our attention to the shape structure of simple representations with repeated parts.

2. Shapes in space

Shapes in space are described logically and philosophically as objects with extension. Theories of extension itself rather than an underlying space in which objects extend have been proposed in many forms. These include Whitehead's theory of extensive connection (Whitehead 1929), Tarski's sphere geometry (Tarski 1956) as well as a wealth of recent formulations for spatial reasoning (for example, Cohn et al (1997), Lemon & Pratt (1997)). A formal comparison of these systems for the plane with point set descriptions (Pratt & Lemon 1998) reveals that expressions with regions have corresponding point set expressions. This indicates that operations on shapes, perhaps in the process of designing, can be represented as computations on point sets. But this only gets the pictures in a visual representation. Their associated interpretation and meaning, formed and constrained by the infrastructure of parts and their relations, is missing.

Shapes are described in an underlying 'space'. Transformations as shape computations can be defined with respect to a coordinate system. Unique descriptions of points and thus of features of shapes, are provided by coordinate systems. These features have unique relationships with the coordinate system. One view is that the coordinate system is a special shape with respect to which the parts of a shape are described. Points are coordinates and lines and planes are equations defining point sets. A square in figure 1, composed of four line segments $x_1, x_2, x_3$ and $x_4$, can be described by coordinates, but visually it is the same as a square of the same size anywhere in the plane – provided of course that there are no other shapes to give context. This is not to say that two squares side by side are the same identical shape.
In a shape, such as figure 2, which is the sum of two squares $a + b$, the two parts $a$ and $b$, are repeated. This shape consisting of two overlapping squares is defined by the relation between the squares. Two squares of the same shape in the same relation anywhere in the plane will give the same shape sum.

3. Equality

The question is - when are two shapes equal? For reference we turn to the familiar notion of the embedding space in which the shapes are located. The squares $a$ and $b$ in figure 2 are equal in the sense that there is a transformation in the underlying space which will match the two identically.

There is a way of looking internally at the shapes $a$ and $b$ themselves to decide if they are equal. Then the concept of shape equality will not depend on transformation. The squares in this case are each a series of relations among lines as shown in figure 1 which indicates perpendicular relations between pairs of lines $(x_1, x_2)$, $(x_2, x_3)$, $(x_3, x_4)$, $(x_4, x_1)$ and equal distance between points at the ends of lines. These are not all independent relations as it is not necessary to specify all the relations on both lines and
points. It is important to note here that the relation is indicated by the picture of the shape in figure 1 but not defined. The picture directs our attention to a possible interpretation in terms of relations but does not define the relation.

Indeed, it may be premature to judge the relations in the shape from the instance presented in the picture of figure 1. For example the relations may be just: 'four lines, no two of which are concurrent'. There are no conditions on endpoints here. If this is the relation then the square is a very particular instance of a more general type, namely a projective quadrilateral. Thus in looking at a shape (and its visual representation) a class of shapes can be associated with an instance. These might correspond to established geometric types such as the example of using projective geometry to create a class of equal 'squares'. However, they may also be defined by shape specific constraints on the particular element and relation in the shape expressed as parametric schema. Parametric shapes and associated grammars are particularly useful in design applications (Shea, Cagan & Fenves 1997, Agwaral, Cagan & Stiny 2000). Freedoms allow specialized variations in the dimensions of features in a shape whilst maintaining overall configuration.

So shapes are equal if properties of parts and their relations are the same. The precise definition will depend on the properties chosen. Shapes themselves are not necessarily objects with complex structure (Earl 1997), however they present a rich background for interpretation. Equality of shapes plays a central role in shape rules and shape grammars where Euclidean geometry is assumed. Euclidean transformations plus scaling are allowed when applying rules. A rule \((a, b)\) replaces shape \(a\) by shape \(b = b_1 + b_2\). It is applied to a shape \(S\) by finding a subshape equal to \(a\) (under Euclidean transformation and scale). Figure 3 shows an example of a rule which is additive. The shape \(a\) is repeated identically by \(b_1\). The rule is not just a symbolic notation since the relative location of \(a\) and \(b\) is critical. The rule thus depends on the spatial relation between shape \(a\) and shape \(b\). The rule specifies a way of changing shape \(S\). Replication of this relation continues to changes \(S\).
4. Spatial relations

Although it not the purpose here to present a formal expression of equality of shapes and equality of relations, we observe that the two ideas are intimately linked. Equality of shape depends on common spatial elements and common relations among these elements. Equality of relations depends on equal shapes. A hierarchy of looking inside shape structures is apparent: equality of shapes depends on equality of relations which in turn depend on equality of shapes and so on. The assumption is that equality of spatial relations can be expressed by equality of shapes. We now indicate how this can be justified.

Two shapes $a$ and $b$ in figure 4 are plane segments. The lines shown in the drawings are boundaries of plane segments, but are not meant to be parts of the shapes. The overall shape areas covered by $a$ and $b$ are the same and $a = b$, noting that $a$ and $b$ may be in different locations. One view, $a$, is specified by equal parts $a_1$ and $a_2$ and the other view, $b$, by two equal parts $b_1$ and $b_2$. All these parts are equal shapes in this example.

Figure 3. Rule $(a,b), b = b_1 + b_2$

Figure 4. Distinct spatial relations, but equal shape sum $a = b$
These relations can be expressed in terms of transformations, $T_a$ and $T_b$, $T_a a_1 = a_2$ and $T_b b_1 = b_2$. The parts $a_1$ and $a_2$ have a different relation from the parts $b_1$ and $b_2$ which is specified by a different transformation. However, what happens if the parts identified are not equal. Figure 5 shows two relations, which can be expressed by transformations. But this is at the expense of introducing more coordinate frames in which to define the transformations (Earl and Krishnamurti 1984). In comparison with figure 4, the spatial relations in figure 5 are equal. The shape $a$ and its parts $\{a', a''\}$ can be transformed to match shape $b$ and its parts $\{b', b''\}$.

![Figure 5. Parts of shapes a and b with equal relations](image)

In figure 4 there was no such ‘transformaton’ which matches parts. The relations were not equal. An apparent difference is that $a'_1, a'_2 = b'_1, b'_2$, but $a_1, a_2 \neq b_1, b_2$. This is a necessary condition for equality of relations, namely that the intersection shapes are equal shapes. However, it is not sufficient as figure 6 demonstrates with parts, intersections and sums all equal, that is, $a''_1 = b''_1$ and $a''_2 = b''_2$, $a''_1 = b''_1$, $b''_2$ and $a''_1 + a''_2 = b''_1 + b''_2$.

![Figure 6. Parts of shapes a and b with unequal relations](image)

The subshapes, $a''_1, a''_2 = b''_1, b''_2$ do not have the same relations in their 'host' shapes $a''_1$ and $b''_2$ (figure 7). They are repeated subshapes of shape $a''_2 = b''_2$. Equality of relations becomes a matter of equality of
relations of subshapes. The critical question becomes: when do repeated subshapes have the same relation with the host shape?

![Figure 7. Spatial relation of subshapes to 'host' shape](image)

5. Repeated Subshapes

Intuitively we expect that symmetry of shapes is implicated. If the host shape has a symmetry then for each subshape there are other repeated subshapes which have the same relation to the host shape. Thus for the square (figure 8), subshapes $c_1, c_2, c_3,$ and $c_4$ all have the same relation in the host shape $S$.

![Figure 8. Repeated subshapes](image)

The number of repetitions (for a subshape which itself has no symmetry) is equal to the order of the symmetry of the host shape. Only a selection of possible repetitions is shown in figure 8. There are no surprises here, but we want to express these conditions in terms of shapes. Equal relations of subshape and host requires that complements are equal: $S - c_1 = S - c_2 = S - c_3 = S - c_4$. This equality of subshapes seems to capture the equality of spatial relations and serves as a possible definition. The particular conditions of equality for shapes, whether by type of geometry
or by parametric constraints will be reflected in the conditions for equality of relations.

Definitions of equality of shape have been given informally. These take account of the different types of equality which depends in turn on the properties we wish to preserve. These may be geometric properties, such as concurrency of lines being preserved as a projective condition, or a special parametric constraint. From equality of shape, an equality of relations is derived. Shape rules depend crucially on the equality of subshapes and applying equal spatial relations as specified in the rules. Stiny (1980, 1982) and Knight (1995) have provided a comprehensive examination of the connections between spatial relations, rules and symmetries.

The purpose of the above informal argument is to indicate how spatial relations and symmetry reduce to considerations of shape equality. The view we explore here is a relativism in which shapes themselves define the objects of visual representation. This can be confusing because diagrams use the spatial relations between symbols to represent meaningful relations between the symbols. This is cited as a powerful advantage of visual representation, in that relations are represented by relations or at least relative positions. Bertrand Russell (1923) puts it like this: "There is a complication about language as a method of representing a system, namely that words which mean relations are not themselves relations".

We make one observation on the connections between symmetry and equality. Recall the shapes in figure 6. Given the shapes \( a''_1 = b''_1, a''_2 = b''_2, a''_1 \cdot a''_2 = b''_1 \cdot b''_2 \), then from an instance of one of the shapes, say \( a''_2 \), the spatial relation can be reconstructed but not uniquely. Figure 6 shows two possibilities. In this case the multiple spatial relations do not derive from symmetries in the parts, intersections or sums, but from the symmetry of the difference \( a''_1 - a''_2 = b''_1 - b''_2 \). This effectively means that it is possible to take the part \( a''_1 \cdot a''_2 \) out of \( a''_1 \) and replace it by \( b''_1 \cdot b''_2 \) to get \( b''_1 \). It is not that equality requires complements to be equal but that uniqueness is jeopardised by the symmetry of complements. Thus two shapes with their sum and intersection defines multiple spatial relations which are determined by symmetries of parts.

The parts of a shape provide a description of the internal structure of a shape. They form the basis for interpretations, including derived features such as boundaries (Earl 1997). The shapes in a closure structure on parts (Stiny 1994) do not seem to define the spatial relations among the parts uniquely. Internal symmetries in the parts account for the multiplicity. The final shape is the same, the parts are the same shapes, but the spatial relations among the parts may be different.
6. Repetition and Symmetry

A shape with repeated parts has a sequence of repetitions in equal spatial relations. These are some of the simplest generative shapes, but appear to show intriguing characteristics. All these are manifestations of symmetry in some way, but the exact relationships remain unclear.

In design, symmetry and repetition can be desirable properties of spatial patterns. The presence of symmetry indicates order and balance. These properties can also be created by applying rules. Perception of symmetry and repetition depends on the ways that shapes are decomposed into parts. These decompositions are readily described by rules (Stiny 1996). The key research in this area of spatial patterns has been conducted by Knight (1995) with subsequent classification of grammars, especially those with simple rules generating repetitive shapes (Knight 1999a,b). The relations between ‘motif, rules and emergent properties are mainly described in terms of symmetry. To complete this paper, a few simple observations are made on the properties of shapes constructed by a repeated spatial relation applied to a single shape.

Suppose a shape $S$ is repeated with a spatial relation $r$ to give the shape $S + rS$. The shape $rS$ has the relation $r$ to the shape $S$. The shape $rS$ is equivalently a transformation of $S$. The result is a shape with two parts $S$ and $rS$. The shape $S + rS$ with its repeated parts may acquire symmetries not present in $S$ or implied by the relation. Figure 9 shows examples of line shapes. Hexagons with one side missing are shown overlaid to indicate relative positions of the repeated parts.

![Figure 9. Sum of repeated shapes S and rS with symmetrical outcome S + rS](image)

There may be several ways to construct the final shape by repetition of a single shape. These multiple decompositions do not seem to yield symmetries in the result. Further if the final shape is symmetrical and there are many decompositions this does not necessarily indicate any symmetry in the original shape. Figure 10 shows a simple example of line shapes.
The addition of shapes is indicated by overlaying two partial octagons to produce the same asymmetrical shape.

![Figure 10](image1.png)

**Figure 10.** Different repetitions giving the same composite shape

We now look more closely at how shapes repeated in different relations yield the same shape sum. Particularly, in the light of multiple repetitions, we would like to know whether these will eventually yield different shapes after many repetitions? Further, do the outcomes display any symmetries? Figure 11 (a) shows an example of repeating parts in different spatial relations which continues to be asymmetrical and give the same shape sum. The original shape is shown bounded by heavy lines and the added shape is shaded. Note here that the shapes are intended to be plane segments. The lines indicating the boundaries are not parts of the shape. Figure 11 (b) is the same at the first repetition but different at subsequent repetitions.

![Figure 11](image2.png)

**Figure 11.** Repeated shapes under different spatial relations (a) with equal shape sums, (b) giving different shape sums at the second and subsequent repetitions

Other cases give the same shape outcomes at one and two repetitions of the spatial relations but are distinct by the third. The example in figure 12 is a symmetric hexagon line shape repeated by different spatial relations. One of the sides of the hexagon is drawn with a weighted line so that the
different spatial relations are apparent. The special weighting should be ignored in the resultant shape sums which are equal in the first and second repetitions but distinct in the third. We note that if the weighted line is included in the shape then although the hexagon shape retains a reflection symmetry, the shape sums are distinct at the first relation.

\[ \text{Figure 12. Repeated shapes under different spatial relations with the same shape sum at one and two repeated relations but distinct shape sums at three relations} \]

A general question arises. How many repetitions of two distinct spatial relations are required to discriminate the relations by their output shapes? This might be considered as an elementary example of spatial reasoning but does not seem to admit an elementary answer. The example in figure 11(b) shows that any number may not discriminate. Another question is suggested. How many repetitions of two distinct spatial relations will determine whether the output shapes will be discriminated? Thus for example with the repeated shapes in figure 11 (a) how many repetitions that do not discriminate shape sums are needed to conclude that repetitions will never discriminate shape sums. For figure 11(b) two repetitions discriminate shape sums. In figure 12 it takes three repetitions to discriminate shape sums. There are similarities here to general questions of surprises in shape computations described by March (1996). Perhaps spatial reasoning is characterised by uncertainty and surprise.

A generation with repeated shapes returns on itself when one of the repeated shapes matches the original shape. The result is a unique shape sum for the repeated shapes. The sum is symmetric as each repeated shape has the same relation to the shape sum. Repeating shapes in specified spatial relations is the subject of symmetry groups of two and three dimensional patterns. Generally these patterns allow more than one spatial relation. Other conditions link together these different spatial relations so
that the pattern of repeated shapes is apparent at all levels of detail – combinations of repeated shapes are also repeated in the same way. Further constraints are imposed by the requirement for a consistent indefinitely repeating 'infinite' pattern. This raises the question of how a spatial relation between repeated shapes can be generalised to other repeated shapes. The relations would not be equal in the sense defined earlier, but they are similar. This is a straightforward application of transformations expressed in terms of relations. Suppose there is a spatial relation \( r \) between repeated shapes \( S \) and \( rS \). For a different shape \( X \) with relation \( w \) to \( S \) we want to repeat shape \( X \) in a relation similar to \( r \). This is represented as a composite relation \( w^{-1}rw \), where \( w^{-1} \) represents the relation of \( S \) to \( X \). The limited number of planar and spatial symmetric 'infinite' patterns arises from the requirement that relations between parts of the pattern at all levels of detail are similar.

Symmetric spatial patterns can be fragments of infinite patterns. However the underlying structure is one of repeated shapes and spatial relations. Other patterns of repeated shapes, not so heavily constrained by similarities of spatial relations among parts, give wider freedom within the discipline of generating according to specified relations. However, these generations present complex spatial properties. Repeated parts can merge in their intersections, hiding differences in spatial relations which might manifest themselves at later stages of generation.

7. Conclusion

Symmetry has been approached generatively through repeated shapes. Shape equality depends on shape relation equality of parts of the shape. In turn equality of relations depends on equality of more detailed parts. The role of particular geometries and specific parametric constraints in shape equality has been discussed. Patterns formed from repeated shapes in repeated single spatial relations are examined. Symmetry of a shape is specified by repeated parts in the same relation to the overall shape. Finally, connections are drawn between symmetries of the original shapes, spatial relations and symmetries of the resulting sums of repeated shapes. Although these are some of the simplest generative shapes it appears to be difficult to predict the properties of these patterns. Distinct relations can yield the same shape sum over some generations and then diverge radically. The paper leaves many open questions on the ways that symmetry works in patterns of repeated shapes.
References


