AN INDUCTIVE CONSTRUCTION OF MINIMALLY RIGID PANEL-HINGE GRAPHS AND APPLICATION TO DESIGN FORM

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Abstract. A panel-hinge framework is a structure composed of rigid panels connected by hinges. It was recently proved that for a so-called generic position, the rigidity of panel-hinge frameworks can be tested by examining the combinatorial property of the underlying graph. In this study, we apply such combinatorial characteristics to create design forms. However, such characterization is only valid for so-called “generic” panel-hinge frameworks. When considering the application of design forms, we need to take into account non-generic cases. In this paper, we develop the method to inductively generate non-generic rigid panel-hinge frameworks consisting of orthogonal panels and to inductively generate rigid panel-hinge frameworks based on fractal geometry coupled with space filling 3-dimensional convex polyhedron as a construction unit. We give examples of forms by the proposed method to demonstrate the applicability to design forms.

Keywords. Panel-hinge framework; Panel-hinge graph; Combinatorial rigidity; Algorithmic design.

1. Introduction

1.1. BACKGROUND AND MOTIVATION

Theory of combinatorial rigidity characterizes the rigidity of frameworks in terms of the underlying graph that represents the connection relationships of joints by members. Combinatorial rigidity does not only provide the
knowledge of the rigidity of frameworks, but also has been applied in a wide range of fields such as mechanical design, molecular dynamics simulations, the development of intellectual CAD and localization of sensor network since the late 90’s. However, the theory of combinatorial rigidity can be applied only to the so-called generic frameworks. We explain the detailed definition of generic rigidity in section 2.1. When considering the applicability to creation of design forms, we need to extend it so as to cope with non-generic case. Designs often feature repeated motifs parallel components, and symmetries (Farre et al., 2013).

This paper focuses on a 3-dimensional panel-hinge framework that consists of 2-dimensional rigid panels connected by "hinges". Here a panel is a planar object such as a board which is rigid (Fig.1(a)). Panels are allowed to move continuously in $\mathbb{R}^2$ so that the relative motion of any two panels connected by a hinge is a rotation around it and the framework is called "rigid" if every motion provides a framework isometric to the original one. We represent a panel-hinge framework as a pair $(G, p)$ of a graph $G = (V, E)$ and a mapping $p$ from $e \in E$ to $p(e)$ in the 2-dimensional affine subspace. Namely, $v \in V$ corresponds to a panel and $uv \in E$ corresponds to a hinge $p(uv)$ which joints the two panels corresponding to $u$ and $v$. Then, $G$ is said to be "realized" as a panel-hinge framework $(G, p)$ in $\mathbb{R}^2$, and is called a "panel-hinge graph".

![Figure 1](image)

*Figure 1. (a) A panel-hinge framework and (b) the panel-hinge graph corresponding to (a).*

Since it is one of the characteristics of algorithmic architecture which is recently proposed by Terzidis (2006) that the structure is constructed by many components, it is expected that combinatorial rigidity will be applied to design form. However, the theory of combinatorial rigidity can be applied only to the so-called generic panel-hinge frameworks. When considering the applicability to creation of design forms, we need to extend it so as to cope with non-generic case.
1.2. THEORETICAL BACKGROUND

Figure 2. (a) A bar-joint, (b) a body-hinge and (c) a panel-hinge frameworks.

A truss structure is a kind of bar-joint framework, which is a structure composed of rigid rods (bars) connected at their ends (free joints) (Fig.2(a)). For a special class of generic 3-dimensional bar-joint frameworks such as generic "body-hinge" and "panel-hinge frameworks" (Fig.2(b) and (c)), a combinatorial characterization was developed by Tay (1989), Whiteley (1988), Katoh and Tanigawa (2011).

**Proposition 1** (Tay, 1989; Whiteley, 1988; Katoh and Tanigawa, 2011) Let $G = (V, E)$ denote a panel-hinge graph (body-hinge graph). Let $\tilde{G}$ denote the graph obtained from a graph $G$ by replacing each edge by 5 parallel edges. $G$ can be realized as a generically infinitesimally rigid panel-hinge framework (body-hinge framework) in $\mathbb{R}^3$ if and only if $\tilde{G}$ contains 6 edge-disjoint spanning trees.

See (Tay, 1989) for a definition of body-hinge graph. For minimally rigid panel-hinge (body-hinge) graphs, Higashikawa et al. (2013) recently proposed that inductive operations that create one with larger size from a smaller one by which we can develop an algorithm for enumerating all minimally rigid panel-hinge simple graphs whose running time is polynomial per output.

The rest of this paper is organized as follows. In Section 2, we introduce necessary notations and facts. In section 3, we introduce the operations that inductively generate non-generic rigid panel-hinge frameworks consisting of orthogonal panels and to inductively generate rigid panel-hinge frameworks based on space filling 3-dimensional convex polyhedron as a construction unit. The operations of section 3 are developed, based on the operations of (Higashikawa et al, 2013). We give the proof that the resulting frameworks are rigid.
2. Preliminaries

2.1. MOTIONS OF BODIES AND HINGE CONSTRAINTS

A "motion" of a panel is a direction-preserving isometry, which is written by a 4 × 4-matrix $M$ with the homogeneous coordinates (see, e.g., (Selig, 2004)). Let us consider two panels $B$ and $B'$ connected by a hinge $H$ through the homogeneous coordinate $p_1 = (p_{1x}, p_{1y}, p_{1z}, 1)$ and $p_2 = (p_{2x}, p_{2y}, p_{2z}, 1)$, and suppose that $M, M'$ are assigned to $B$ and $B'$, respectively. Then, the hinge constraint by $H$ is defined as $M \ p_1 = M' \ p_1$ and $M \ p_2 = M' \ p_2$. Taking the derivative of these equalities (as $M$ and $M'$ vary continuously), we have

$$I \ p_i = I' \ p_i \quad \text{for } i = 1, 2$$

where we can regard $I$ and $I'$ as infinitesimal motions assigned to $B$ and $B'$. It is known that $I$ is represented by $\begin{pmatrix} R & v^T \\ 0 & 0 \end{pmatrix}$ by using a 3 × 3-skew symmetric matrix $R = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$ and $v = (v_x, v_y, v_z)$ (see, e.g., (Selig, 2004)). Similarly, $I'$ is represented as $\begin{pmatrix} R' & v'^T \\ 0 & 0 \end{pmatrix}$.

It is easy to observe that this equation is satisfied if and only if there exists $t \in \mathbb{R}$ such that

$$(\omega - \omega', v - v') = I(\begin{pmatrix} p_{1x} & 1 \\ p_{2x} & 1 \end{pmatrix}, -\begin{pmatrix} p_{1y} & 1 \\ p_{2y} & 1 \end{pmatrix}, \begin{pmatrix} p_{1z} & 1 \\ p_{2z} & 1 \end{pmatrix}, \begin{pmatrix} p_{1y} & p_{1z} \\ p_{2y} & p_{2z} \end{pmatrix}),$$

where $\omega = (\omega_x, \omega_y, \omega_z)$. This implies that, if we identify the infinitesimal motions $I$ and $I'$ with the 6-dimensional vectors $s = (\omega, v)$ and $s' = (\omega', v')$, the hinge constraint $s - s'$ of $H$ to be proportional to by the so-called 2-extensor, that is, $C(H)$ supporting $H$ (Katoh and Tanigawa, 2011).

By the definition of infinitesimal motions, taking any basis $r_i(p(e))$ ($1 \leq i \leq 5$) of the orthogonal complement of the vector space spanned by $C(p(e))$, we can say $S$ is an infinitesimal motion of $(G, p)$ if and only if $(S(u) - S(v)) \cdot r_i(p(e)) = 0$ for all $i$ with $1 \leq i \leq 5$ and for all $e = uv \in E$. An infinitesimal motion $S$ is called "trivial" if $S(u) = S(v)$ for all $u, v \in V$, and $(G, p)$ is said to be "infinitesimally rigid" if all infinitesimal motions of $(G, p)$ are trivial. Hence, the constraints to be an infinitesimal motion are described by $5|E$.
linear equations over $S$. In other words, $S$ is infinitesimal motion of $(G, p)$ if and only if it is in the null space of a $5|E| \times 6|V|$-matrix $R(G, p)$ written as

$$
R(G, p) = e_{i=1}^{n} \begin{pmatrix} 
\cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r(p(e)) & \cdots & -r(p(e)) & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
$$

where $r(p(e))$ denotes a $5 \times 6$-matrix defined by $r(p(e)) = \begin{pmatrix} r_1(p(e)) \\
\vdots \\
r_5(p(e)) \end{pmatrix}$. We call $R(G, p)$ the "rigidity matrix" of $(G, p)$. $(G, p)$ is "infinitesimally rigid" if and only if the rank of $R(G, p)$ is exactly $6(|V| - 1)$.

$(G, p)$ is generic if $R(G, p)$ and all of its edge-induced submatrices have maximum rank, taken over all 3-dimensional panel-hinge realizations of $G$. For generic frameworks, rigidity is equivalent to infinitesimal rigidity (Asimow and Roth, 1978; Whiteley, 1988).

2.2. CHARACTERIZATION OF GENERIC RIGID PANEL-HINGE FRAMEWORKS

In the previous subsection, we showed that testing whether a panel-hinge framework is rigid or not is reduced to the rank computation of the rigidity matrix. However, as the number of components of the framework increases, calculating the rank of the rigidity matrix of the framework becomes computationally difficult and may cause numerical errors. Therefore, it is meaningful to consider the generic panel-hinge frameworks. Katoh and Tanigawa (2011) proved that the generic rigidity of minimally rigid "panel-hinge framework" (Fig. 2(b)) is equivalent to that of body-hinge frameworks.

3. Application to Design Form

3.1. GENERATION OF FORM

In this section, we propose the operations that inductively generate rigid panel-hinge frameworks that consist of non-generic hinge configuration based on the operations of (Higashikawa et al, 2013). We first introduce the method to inductively generate rigid panel-hinge frameworks such that every panel is a rectangle and is parallel to one of $xy$-, $yz$- and $zx$-plane, and a hinge is attached to an edge of a panel. We define two inductive operations which construct a rigid panel-hinge framework of a larger size. Furthermore, we show the examples that are obtained by applying the proposed operations. As already mentioned, the hinge configuration of panels is not generic. There-
fore, we will prove that the resulting frameworks obtained by applying operations are rigid.

Operation 1 (Add 2-panel): Choose the panel \( P_1 \) where one of the endpoints \( v \) of the panel is not connected to other panels. Then, add two new panels \( P_2 \) and \( P_3 \) to \( P_1 \) such that \( P_1, P_2 \) and \( P_3 \) are orthogonal to each other in such a way that \( P_2 \) and \( P_3 \) are connected to \( P_1 \) by hinges and \( P_2 \) is connected by \( P_3 \) by a hinge so that three new hinges meet at \( v \).

Operation 2 (Add 3-panel): Choose the panel \( P_1 \) where one of the endpoints \( v \) of the panel is not connected to other panels. Then, add three panels \( P_2, P_3 \) and \( P_4 \) to \( P_1 \) such that \( P_1, P_2, P_3 \) and \( P_4 \) are orthogonal to each other in such way that \( P_2 \) and \( P_4 \) are connected to \( P_1 \) by hinges, \( P_2 \) is connected by \( P_3 \) and \( P_3 \) is connected by \( P_4 \) by a hinge so that four new hinges meet at \( v \).

3.1.1. Operation Add 2-panel

Figure 3. Illustration of Operation Add 2-panel ((a) panel-hinge frameworks and (b) their corresponding panel-hinge graphs).

Figure 3 shows a panel-hinge framework generated by Add 2-panel operation. Figure 4 shows examples that are obtained by applying this operation several times.

We now give the proof that if the framework is rigid, the resulting framework obtained by adding two panels is rigid. Let \( P_1 = (x_1, y_1, 0), P_2 = (x_1, y_2, 0), P_3 = (x_1, y_2, z_1) \) and \( P_4 = (x_2, y_1, 0) \) with \( x_1 \neq x_2, y_1 \neq y_2 \) and \( z_1 \neq 0 \). Let 2-extensor of hinges \( P_1P_2, P_2P_3 \) and \( P_2P_4 \) be \( C(p(1)) \), \( C(p(2)) \) and \( C(p(3)) \) respectively. By (2), we have

\[
\begin{align*}
C(p(1)) &= (0, -(y_1-y_2), 0, 0, 0, x_1y_2-x_1y_1), \\
C(p(2)) &= (0, 0, -z_1, y_2z_1, -x_1z_1, 0), \\
C(p(3)) &= (x_1-x_2, -(y_2-y_1), 0, 0, x_1y_1-x_2y_2).
\end{align*}
\]
Let the infinitesimal motion of the panels connected by hinges \(P_1P_2, P_2P_3\) and \(P_2P_4\) be \(S_1\) and \(S_2\), \(S_2\) and \(S_3\) and \(S_3\) and \(S_1\) respectively. By (2), we have 
\[S_2-S_1 = t_1C(p(1)), S_3-S_2 = t_2C(p(2))\] and \(S_1-S_3 = t_3C(p(3))\). Then, we have 
\[0 = t_1C(p(1)) + t_2C(p(2)) + t_3C(p(3)).\] (6)

By (3), (4), (5) and (6), we have 
\[0 = t_3(x_1-x_2), 0 = (y_1-y_2)(t_3-t_2), 0 = -t_2z_1, 0 = t_1(x_1y_1-x_1y_2) + t_3(x_1y_1-x_2y_2).\] By \(x_1 \neq x_2, y_1 \neq y_2\) and \(z_1 \neq 0\), we have \(t_1 = t_2 = t_3 = 0\). Then, by (6), (7) and (8), we have \(S_1 = S_2 = S_3\). Therefore, the framework obtained by adding this operation is also rigid.

Figure 4. The examples that are obtained by repeatedly applying Add 2-panel operations.

3.1.2. Operation Add 3-panel

Figure 5 shows a panel-hinge framework generated by Add 3-panel operation. Figure 6 shows the example that is obtained by applying this operation several times.

We now give the proof that if the framework is rigid, the resulting framework obtained by adding three panels is rigid. Let \(P_1 = (x_1, 0, 0), P_2 = (x_2, 0, 0), P_3 = (y_1, 0, z_1), P_4 = (y_2, 0, z_1), P_5 = (0, y_1, 0), P_6 = (0, y_2, 0)\) with \(x_1 \neq x_2, y_1 \neq y_2\) and \(z_1 \neq 0\) (Figure 10). Let 2-extensor of hinges \(P_1P_2, P_2P_3, P_3P_4\) and \(P_4P_5\) be \(C(p(1)), C(p(2)), C(p(3))\) and \(C(p(4))\) respectively. By (2), we have 
\[C(p(1)) = (x_1-x_2, 0, 0, 0, 0, 0), C(p(2)) = (x_2-x_1, 0, 0, 0, 0, 0), C(p(3)) = (0, -y_1-y_2, 0, 0, 0, 0)\] and \(C(p(4)) = (0, -y_1-y_2, 0, 0, 0, 0)\).
Let the infinitesimal motion of the panels connected by hinges $P_1P_2$, $P_2P_3$ and $P_2P_4$ be $S_1$, $S_2$, $S_3$, $S_4$ and $S_5$ respectively. By (2), we have $S_2 - S_1 = t_1 C(p(1))$, $S_3 - S_2 = t_2 C(p(2))$, $S_4 - S_3 = t_3 C(p(3))$ and $S_1 - S_4 = t_4 C(p(4))$. Then, as in the same manner of Operation Add 2-panels we can show that $S_1 = S_2 = S_3 = S_4$. Therefore, the framework obtained by adding three panels is also rigid.

Figure 6. The examples that are obtained by repeatedly applying Add 3-panel operations.

### 3.1.3. Generation of panel-hinge frameworks based on space-filling convex polyhedron and fractal geometry

The fractal geometry is the geometry which has the recursive structure (Mandelbrot, 1983). We introduce the method which inductively generates fractal panel-hinge frameworks where we use the cube and the truncated octahedron of space-filling convex polyhedron as the basic unit. It is possible to generate panel-hinge frameworks without overlapping by using space-filling polyhedron. In this case, although the details are omitted, we checked...
the rigidity of the initial framework $F_0$ which is illustrated at the leftmost position in Figures 7(a) and 8(a) by the same method used in Section 3.1.2. Then, we replace each panel of a panel-hinge framework $F_0$ by $F_0$ itself which produces panel-hinge framework $F_1$. Similarly, we obtain $F_2$ from $F_1$. Therefore, all panel-hinge framework newly constructed has the same panel-hinge graph where each vertex corresponds to the previous framework. Figures 9(a) and (b) show the examples that are recursively generated by the connection relationships of the graph of Figure 9(c). Thus, it is possible to generate various rigid frameworks by this method. Then, we generate 3D models by using Rhinoceros of a 3D CAD software and Python of a scripting language.

![Figure 8](image_url)

*Figure 8. (a) The examples that are recursively generated with the truncated octahedron as the base unit and (b) the panel hinge graph corresponding to the leftmost framework of (a).*

![Figure 9](image_url)

*Figure 9. (a), (b) The examples that are recursively generated with the truncated octahedron as the base unit and (c) the panel hinge graph corresponding to the framework of (a) and (b).*

### 3.2. FABRICATION OF THE MODEL

We create the model by using wood (Figure 10). Since panels are connected by hinges, it is easy to unroll the framework. Therefore, we consider that it is possible to use this model as the temporary structure.
4. Conclusion

Based on the operations that inductively generate rigid panel-hinge graphs in (Higashikawa et al, 2013), we develop the method to inductively generate non-generic rigid panel-hinge frameworks consisting of orthogonal panels and to inductively generate rigid panel-hinge frameworks based on space filling 3-dimensional convex polyhedron as a construction unit. We generated forms by proposed operations and demonstrated that it is possible to apply the proposed operations to design form.

References


