A Model of Land use Conversion and Its Application

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ABSTRACT

A quantitative model for analyzing the spatial distribution of land use utility is proposed. This model is based on the random bidding theory in which location behavior is decided according to the size of utility to be obtained. The utility function used here consists of the benefit and the cost in the process of land use transition. The benefit is described as the positive utility that can be obtained by doing the corresponding land use at the place. The cost is described as the negative utility that is necessary for changing the land use from one state to the others. The most likelihood method is generally employed to estimate the parameters of this kind of models. However, we attempt to propose the other statistical method through the mathematical consideration. Using this model, it is possible to obtain the spatial distribution of land use utility that differs with the places and with the land use states. We can also evaluate the effects of a change of land-price or construction costs on our utility. Namely, our location behavior can be estimated numerically relating with the social or economic factors. As numerical examples, we apply the proposed model to the actual land use data and access the effectiveness of the model.

1 INTRODUCTION

In a field of city / regional planning, it is one of the important subject of concern to predict a trend of land use conversion. Therefore, many mathematical or numerical models have been developed. Much of these works are reviewed, for example, in Batty (1967), Lowry (1968), Kilbridge (1969) and Foot (1981).

Macroscopic analysis using the aggregated value of each consumer's income or company's capital requires us to use only a few variables for describing the objective phenomenon. However, there remains uncertain relationship between the aggregated value and each individual economy behavior. On the other hand, microscopic analysis can describe the phenomena theoretically and precisely. However it is often difficult to apply to the actual objects because of the limitation of data.

Econometric models are considered as the former and urban economic models are the later. Urban models are between them. This is because the urban models have been developed from the macroscopic models to the microscopic models by improving the assumptions to fit to the observed values. One of the main ideas employed from the microscopic analysis is the assumption of utility maximizing behavior. Although almost urban models are aggregation models, this assumption is combined into most urban models implicit or explicitly.

In the past several years, some location behavior models based on the random utility theory have been developed considering the variety of individual behavior. Nakamura et all (1981) proposed a land use model for suburban area and Hayashi et al (1984) proposed an industrial location model. Morisugi et al (1984) constructed a method for forecasting the residential behavior and evaluating the environmental benefit. This tendency is enlivened by increasing availability of computer facilities and associated software, especially by developments of Geographical Information Systems (GIS).

Wegener (1994) says that almost all the land use models, which have been developed in the resent years, are based on the random utility theory. When the random utility theory is employed into land use models, we can take two different approaches (Miyamoto 1987). One is an approach in which we discuss the location probability of each candidate place under the condition of fixed location subject. In the other approach, we discuss the location probability of each location subject under the condition of fixed candidate place. The former is called a random utility model. The latter is called a random bidding model, and there are studies such as Ellickson (1981), Lerman and Kern (1983), Kashiwaya and Ogura (1986) and Ando (1987). Also some studies attempt to integrate these two approaches, for example, Miyamoto and Kitazume (1989).

In this paper, we attempt to describe the location behavior of urban activities based on the latter approach and propose a model to measure the value of land use utility and its spatial distribution. Furthermore using the actual land use data we analyze the spatial distribution of land use utility and examine the effectiveness of our model. With some numerical examples, we show that how a change of land-price or construction costs effects on our utility, and how the location behavior of land use can be changed.

2 MODELING OF LAND USE TRANSITION

Owners of lands can change the attribute of lots by rearranging the land or reconstructing buildings. Landowners will do it not only for own necessity as users but also for selling or for lending. In any case, the owner evaluates the effect of land improvement and decides his concrete behavior. On the other hand, each user reevaluates the land, improved by the investment of the owner, and puts the bid price to the land. In the process of a new bid price settlement, one can be a new land user or owner if he/she can propose the highest bid price. Thus, the landowner does various investments and actions for getting the highest land rent from a user. Hence, we can consider that the owner is maximizing the following function of utility,

$$\tilde{U}_{ij}(r) = u(K_{ij}(r), R_i(r)) \qquad (1)$$

where, i (i=1,...,n) or j (j=1,...,n) represents a state of a land. $\tilde{U}_{ij}(r)$ represents the utility that can be obtained by changing the land use state from j to i at the place r.

$K_{ij}(r)$ denotes the improvement cost for changing the state of land use. $R_i(r)$ is the expected benefit when the land use state is changed to *i* at the place *r*. Thus, the utility maximization by landowners leads to the conversion of land use.

Assume that the utility $\tilde{U}_{ij}(r)$ can be divided into a deterministic part $U_{ij}(r)$ and a stochastic part $e_{ij}(r)$. If we assume the distribution of $e_{ij}(r)$ is Gumbel distribution, the utility maximizing theory gives a so-called logit model as follows;

$$P_{ij}(r) = \frac{\exp[U_{ij}(r)]}{\sum_k \exp[U_{kj}(r)]} \tag{2}$$

where $P_{ij}(r)$ is the transition probability of land use change from *j* to *i* at a place *r*.

Next, we formulate a function of utility $U_{ij}(r)$ obtained in the process of land use transition. Several kinds of functions are available to be employed. We consider here the simple linear function as follows;

$$\begin{aligned} U_{ij}(r) &= b_1 R_i(r) - b_2 K_{ij}(r) \\ &= U_i(r) - C_{ij}(r) \end{aligned} \tag{3}$$

where b_1 and b_2 are positive constants. The benefit, denoted by $U_i(r)$, is therefore the utility obtained by doing land use *i* in a place *r*, which is equivalent to the value of bid price in term of utility. The cost, denoted by $C_{ij}(r)$, is the utility that is necessary for changing the state of land use from *j* to *i*, which is equivalent to the value of improvement cost in term of utility.

Furthermore, we assume that $C_{ij}(r)$ can be represented as follows;

$$C_{ij}(r) = v_i + u_j + c(r) + c_o \quad \text{for } i \neq j, \tag{4}$$

$$C_{jj}(r) = 0 \tag{5}$$

where v_i is the cost depending on the destination state of land use *i*, and u_j is the cost depending on the original state of land use *j* before a change. Also, $c(r)$ is the cost depending on the land-price at the place *r*, and c_o denotes the common initial cost of land use change. Furthermore, the following assumption is made to simplify this model;

$$u_j = m_j v_j \quad \text{for all } j, \tag{6}$$

where m_j is a positive constant. Considering equations (3), (4) and (5), the utility $U_{ij}(r)$ obtained by land use transition can be rewritten in the following form;

$$U_{ij}(r) = U_i(r) - v_i - m_j v_j - c(r) - c_o \quad \text{for } i \neq j\,, \tag{7}$$

$$U_{jj}(r) = U_j(r)\,. \tag{8}$$

3 ESTIMATION METHOD OF PARAMETERS

3.1 **Method for Parameter Estimation**

Most likelihood method is generally employed for estimating the parameters of logit models. However, it is difficult to estimate $U_i(r)$ and v_i in equation (7) separately. Hence we estimate parameters by least square method to avoid this problem.

The following equation is obtained from equations (2), (7) and (8);

$$\ln Q_{ij}(r) = -(1+m_i)v_i - (1+m_j)v_j - 2c(r) - 2c_o \quad \text{for } i \neq j\,, \tag{9}$$

where $Q_{ij}(r) = \dfrac{P_{ij}(r)}{P_{jj}(r)} \cdot \dfrac{P_{ji}(r)}{P_{ii}(r)}$.

Assume that the cost $c(r)$ can be obtained by multiplying a positive constant parameter a to the actual land-price, denoted by $b(r)$. Using dummy variables d_k $(k = 1,\ldots,n)$ for expressing the land use states, equation (9) can be rewritten into the following form;

$$\ln Q_{ij}(r) = -\sum_k (1+m_k)v_k d_k - 2ab(r) - 2c_o \quad \text{for } i \neq j\,, \tag{10}$$

where $d_k = \begin{cases} 1 & \text{for } k = i \text{ or } j \\ 0 & \text{otherwise} \end{cases}$.

When we consider the variable d_k $(k = 1,\ldots,n)$ and $b(r)$ as the explanatory variables and the others as the regression coefficients, equation (10) is equivalent to a multiple regression model. However, we have to discuss the following two problems before the estimation of parameters by multiple regression analysis.

The first problem is on a distribution function of error in this model. That is, in a multiple regression model, we assume a condition in which the error distribution is normal. In a logit model, however, Gumbel distribution is assumed. This contradiction between these assumptions must be discussed. The second problem is on the collinearity that exists in the dummy variables. That is, the sum of value of dummy variables d_k over k is a constant. We discuss these two problems in the following.

3.2 **Problem of Distribution Function**

Denote the error in the multiple regression model of equation (10) by e. It can be expressed in the form;

$$e = e_{ij}(r) - e_{jj}(r) + e_{ji}(r) - e_{ii}(r). \tag{11}$$

The distribution of each term of right hand side is assumed to be Gumbel distribution. However, if we assume that $e_{ij}(r)$ and $e_{jj}(r)$ are independent Gumbel distribution whose prescribing parameters (h_j, w) are the same, the difference between $e_{ij}(r)$ and $e_{jj}(r)$, denoted by e_j, becomes the logistic distribution;

$$F(e_j) = \frac{1}{1 + \exp[-we_j]}. \tag{12}$$

Details are described in the appendix. If we rewrite the parameter w to $\frac{4}{\sqrt{2p}s}$ where s denotes the standard deviation of normal distribution, the distribution of difference e_j is very close to normal distribution $(0, s^2)$. In the same way, difference between $e_{ji}(r)$ and $e_{ii}(r)$, denoted by e_i, can be also considered approximately as normal distribution. Thus we can consider that the distribution of error e is approximately the same as normal distribution, since the linear sum of independent normal distribution makes normal distribution.

3.2 **Problem of Collinearity**

The dummy valuable d_k, which is explanatory variable in the regression model, has the following property;

$$\sum_k d_k = 2 \;. \tag{13}$$

Hence we can not estimate the value of parameter directly by the regression analysis. Then we forces on the cost v_i that depends on the destination land use. The cost v_* $(i = *)$ necessary for vacant lot can be considered very small comparing to the other cost v_i $(i \neq *)$. Then we assume that

$$v_* \cong 0 . \tag{14}$$

Namely, a problem of collinearity can be avoided if the dummy variable d_* corresponding to v_* is omitted from the explanatory variables.

4 ESTIMATION METHOD OF LAND USE UTILITY

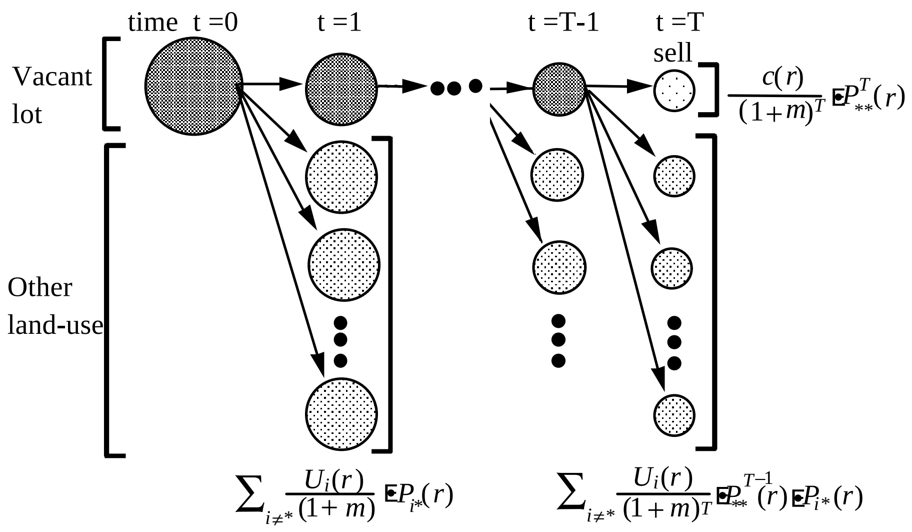
Using equations (2), (7) and (8), the difference of utility between $U_i(r)$ and $U_j(r)$ is represented as follows;

$$\begin{aligned} D_{ij}(r) &= U_i(r) - U_j(r) \\ &= \frac{1}{2}\left\{ \ln \frac{P_{ij}(r)}{P_{jj}(r)} \cdot \frac{P_{ii}(r)}{P_{ji}(r)} + (1-m_i)v_i - (1-m_j)v_j \right\} \quad \text{for } i \neq j . \end{aligned} \tag{15}$$

The above equation shows that we can get the difference between the values of utility of two land use states, *i* and *j*, if we know the profile of m_i (i=1,...,n). However, only the difference of utility can be obtained, i.e., we can't get the value of $U_i(r)$ itself. Then we discuss the method for estimating the value of $U_i(r)$ by paying attention to the utility $U_*(r)$ of vacant lot.

The vacant lot can be regarded as a transient state from one land use state to the others in the transition process. Then we can consider that the utility of vacant lot in the present time can be equivalent to the expected value of utility that will be obtained after the transition in the future (see figure 1).

Figure 1: **Utility of Vacant Lot**



Assume that the utility $U_i(r)$ is constant on time, and the value of utility obtained by selling the land is equal to $c(r)$. Assuming the vacant lot will be sold or changed to the other states by time *T*, the utility $U_*(r)$ of the vacant lot is represented as follows;

$$
\begin{aligned}
U_*(r) &= \sum_t^T \sum_{i \neq *} \frac{U_i(r)}{(1+m)^t} \cdot P_{i*}(r) \cdot P_{**}^{t-1}(r) + \frac{c(r)}{(1+m)^T} \cdot P_{**}^T(r) \\
&= S_1(m,T) \cdot \sum_{i \neq *} U_i(r) \cdot P_{i*}(r) + S_1(m,T) \cdot c(r) \qquad (16)
\end{aligned}
$$

where $S_1(m,T) = \sum_t^T \frac{P_{**}^{\;t-1}(r)}{(1+m)^t}$, $S_2(m,T) = \frac{P_{**}^{\;T}(r)}{(1+m)^T}$ and *m* is a positive constant of marginal efficiency of capital.

On the other hand, equation (15) shows that $U_i(r)$ can be described by using $U_*(r)$ as follows;

$$
U_i(r) = D_{i*}(r) + U_*(r) \quad \text{for } i \neq * . \qquad (17)
$$

From equations (16) and (17), we can get the value of the utility of vacant lot as follows;

$$
U_*(r) = \frac{S_1(m,T) \cdot \left\{ \sum_{i \neq *} D_{i*}(r) \cdot P_{i*}(r) \right\} + S_2(m,T) \cdot c(r)}{1 - S_1(m,T) \cdot (1 - P_{**}(r))} . \qquad (18)
$$

Thus the value of utility of the other land use states can also be estimated from equation (17).

5 A CASE STUDY OF THE METROPOLITAN AREA

5.1 **The Study Area**

We attempt to analyze numerically the utility of land use through the appreciation of the proposed model to the actual land use data of Tokyo Metropolitan area. The data used here are the *Detailed Digital Land use Data* compiled by the Geographical Survey Institute, Ministry of Construction Japan. Land use data of ten meters wide square grid cell (1979 and 1984), the time distance data and public announcement price of land (1979) are used.

The land use utility can be different according to the accessibility to the central business district (CBD). In many studies, the accessibility was described by the Euclidean distance from the CBD, because of data limitation or for the models' simplicity. As Brigham (1965) suggested, however, distance from the CBD is an

Figure 2: **Study Area**

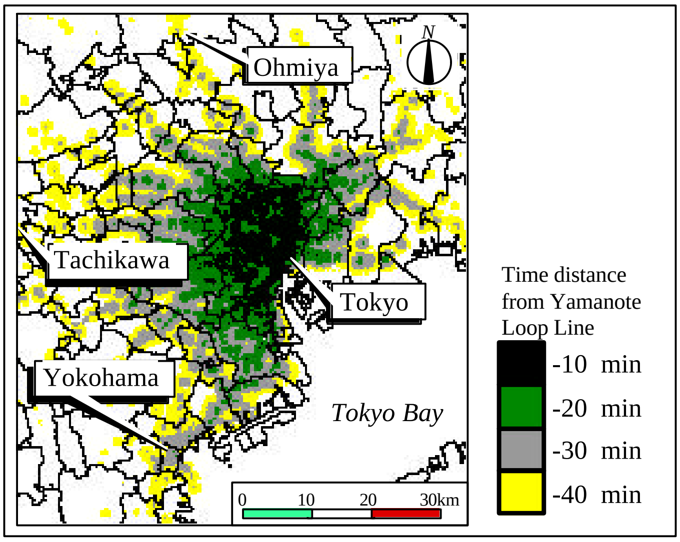


Table 1: **Land use State**

NO.	RE-CLASSIFIED STATE	ORIGINAL STATE
1	forest or field	forest, wasteland, rice field, field
*	vacant lot	land under development, vacant lot
3	industrial lot	industrial lot
4	regular residential lot	regular residential lot
5	densely residential lot	densely residential lot
6	high-rise residential lot	high-rise residential lot
7	commercial lot	commercial lot
8	public space	road, park, green area, public facility
	Others	river, lake, swamp, sea

imperfect proxy for accessibility, because transportation services vary widely between sites located in the same distance from the CBD. Then the time distance zone r is established according to the time distance to the Yamanote Loop Line for five minutes (see figure 2). Land use states are re-classified to get the stable transition probability $P_{ij}(r)$ (see table 1).

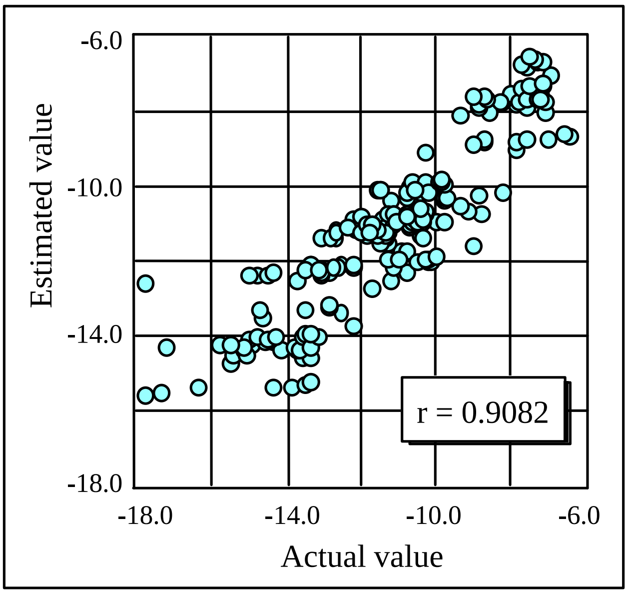
5.2 **Fitness of The Model**

Parameters of equation (10) are estimated by multiple regression analysis through the above procedure. The results are shown in table 2 and figure 3. The correlation coefficient between the estimated values and the actual values of $\ln Q_{ij}(r)$ shows good fitness of the model. All estimated values are significant at a =0.001 level (two-tailed test)

Table 2: **Estimated Parameters**

LAND USE STATE	REGRESSION	COEFFICIENTS	STANDARDIZED VALUE
1. forest or field	$(1+m_1)v_1$	4.6467	0.7646
3. industrial lot	$(1+m_3)v_3$	3.6873	0.6389
4. Regular residential lot	$(1+m_4)v_4$	3.3593	0.5896
5. densely residential lot	$(1+m_5)v_5$	6.6967	1.1448
6. high-rise residential lot	$(1+m_6)v_6$	3.5874	0.6257
7. commercial lot	$(1+m_7)v_7$	2.5759	0.4548
8. public space	$(1+m_8)v_8$	3.2890	0.5808
9. land-price (million yen)	a	0.7147	0.0802
10. common initial cost	c_o	1.9230	1.9230

Figure 3: **Result of Multiple Regression**



Furthermore, the marginal efficiency of capital is set in m=0.1 here, and we assume that $m_i = 0.5\,(\text{for all } i)$ and the vacant lot will be sold or changed into the other land use by time T=3. The land use utility $U_i(r)$ and land use transition probabilities $P_{ij}(r)$ are estimated from equation (2), (17) and (18) using the estimated parameters in table 2. The correlation coefficients between the estimated values and the actual values of $P_{ij}(r)$ are shown in table 3. We can say that the fitness of this model is very good.

Table 3: **Fitness of Transition Probability Model**

TIME DISTANCE	$i = j$	$i \neq j$
$r = 5$	0.9620	0.8564
$r = 10$	0.9812	0.9458
$r = 15$	0.9196	0.8645
$r = 20$	0.9729	0.9151
$r = 25$	0.9609	0.9390
$r = 30$	0.9535	0.9009
$r = 35$	0.9260	0.8781
$r = 40$	0.9477	0.8794
total area	0.9345	0.8871

5.3 **Distribution of Land use Utility**

The spatial distribution of estimated values of utility $U_i(r)$ are plotted in figure 4. The utility $U_7(r)$ of commercial lot is very high in the CBD, however, it decreases sharply according to the increase of distance *r*. On the other hand, the decrement of utility $U_4(r)$ of regular residential lot is gradual comparing to the other land use utility. It reverses with the utility $U_6(r)$ of high-rise residential lot in a place of approximately thirty-two minutes from the CBD.

Figure 4: **Spatial Distribution of Land Use Utility**

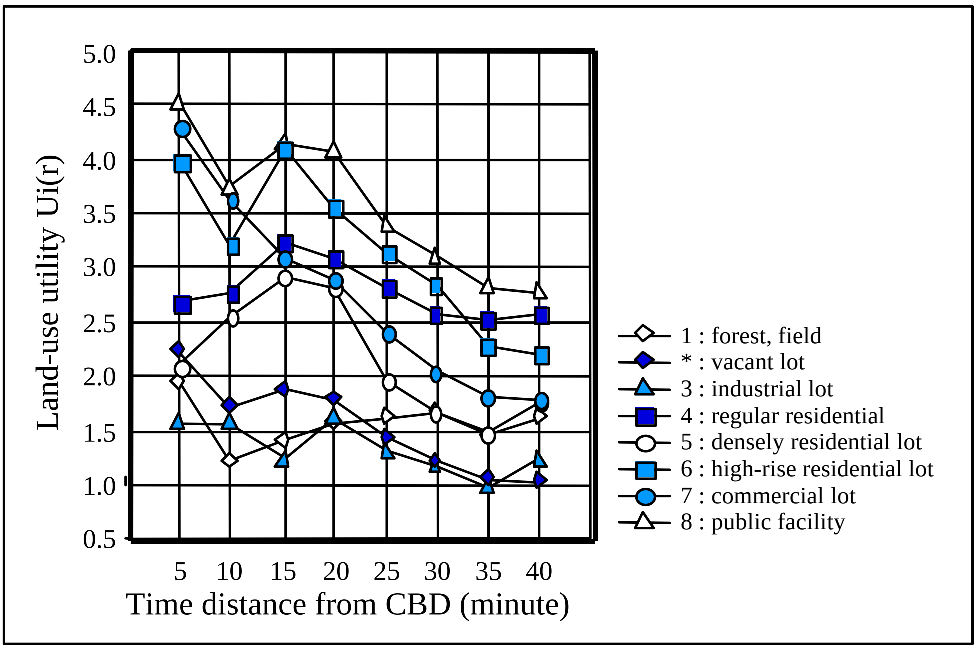
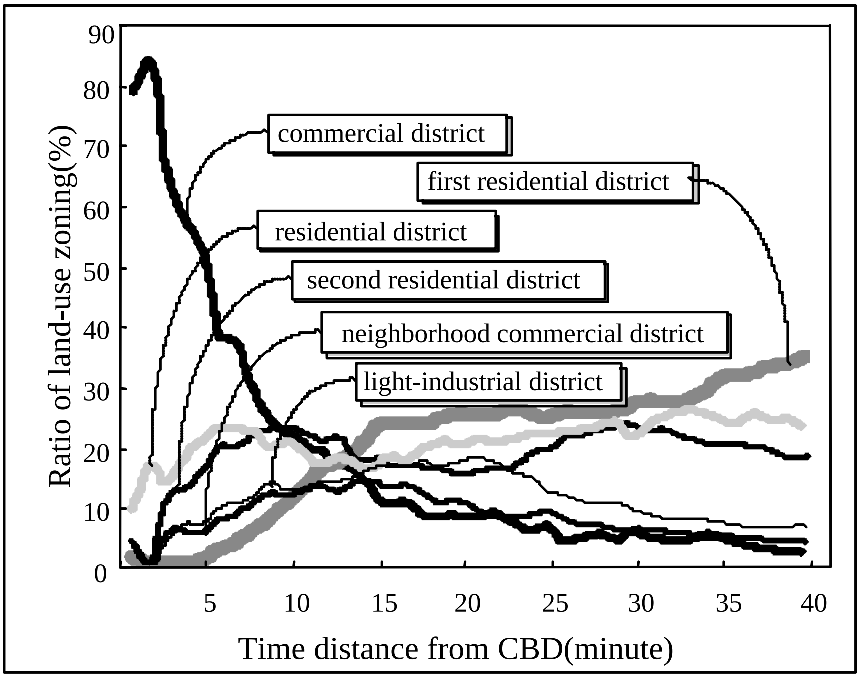


Figure 5: **Distribution of Land Use Zoning**



Although the land use utility has been considered to be much higher in a place near by the CBD (Alonso, 1964), figure 4 shows that all three kinds of residential lot have a peak of utility in a place of approximately fifteen minutes from the CBD.

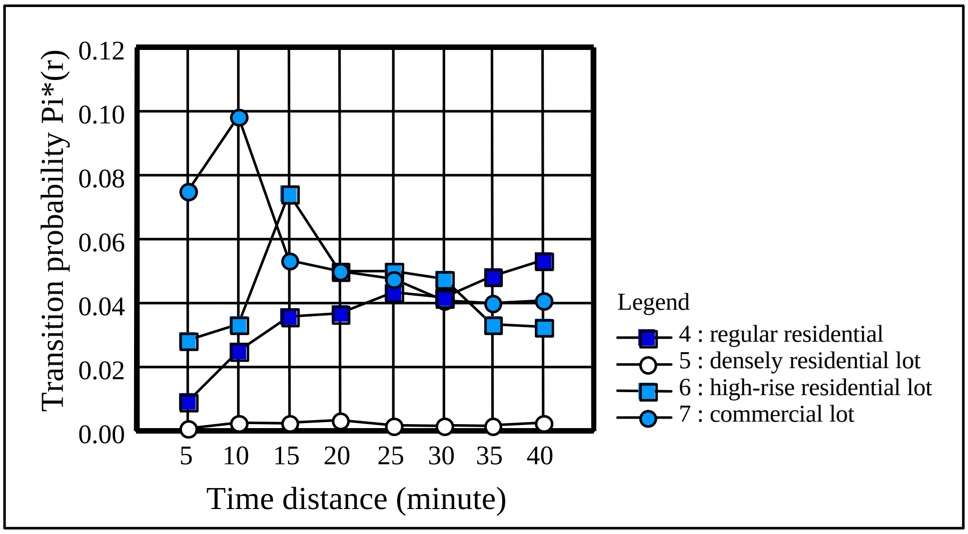
We compare this property with a spatial distribution of land use zoning (see figure 5). We can see that various land use zonings are coexisting near the place of approximately fifteen minutes. On the other hand, the CBD is specialized in a commercial use. It is very interesting that the utility for residence is rather reduced in the CBD specialized in a commercial use. This fact indicates that the land use mixture might produce the benefit for the residence.

6 SIMULATIONS OF LAND USE TRANSITION

The transition probability of land use depends not only on land use utility $U_i(r)$ but also on the transition cost $C_{ij}(r)$. In the following case studies, we pay attention to the land use transition from vacant lot, and attempt to examine the behavior of transition probability when the transition cost $C_{ij}(r)$ will change.

The present transition probability from the vacant lot is shown in figure 6. Since the land-price is so high in the CBD, the probability staying as vacant lot is very high and the transition probability to other land use states are low.

Figure 6: **Transition Probability from Vacant lot**



Transition probability from vacant lot to regular residential lot increases according to the distance *r*. It reverses with that of high-rise residential lot or commercial lot at the place of approximately thirty minutes. Thus, these three land use are contended with each other in a place of thirty minutes from the CBD.

As numerical examples, we attempt to make some simulations of land use transition using the proposed model with estimated parameters. As a first simulation, we consider the case that the cost v_4 of regular residential lot decreases ten percent. The result is shown in figure 7. The transition probability from vacant lot to regular residential lot is much influenced in suburbs than near the CBD. The reversal place with high-rise residential lot and commercial lot is moved toward the CBD about five minutes.

As a second simulation, the land-price $b(r)$ is assumed to be doubled. The transition probabilities from vacant lot to other land use are shown in figure 8. Land use transition probability from vacant lot receives larger influence in the CBD. For instance, comparing figure 8 with figure 6, land use transition probability in a place of five minutes is reduced to half. This behavior shows that vacant lot will remains and activities of land use transition will be reduced.

Figure 7: **Effects of Construction Cost**

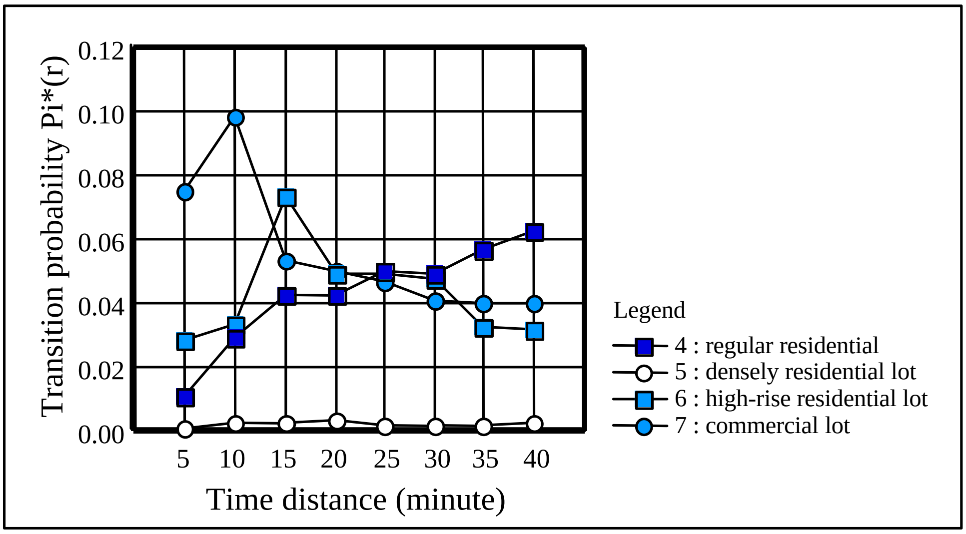
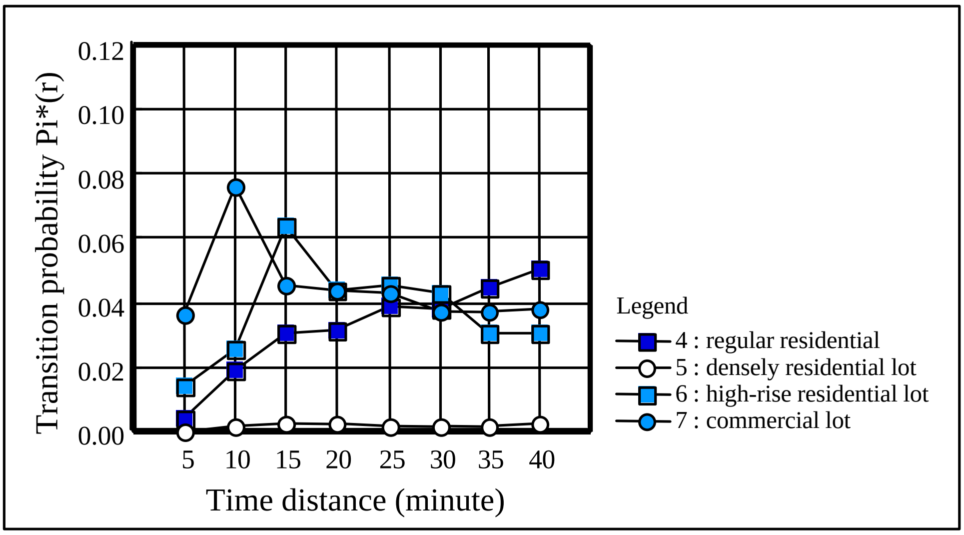


Figure 8: **Effects of Land-price**



7 CONCLUSION

The structure of land use transition is described using a concept of land use utility. Using the proposed model we can obtain the spatial distribution of the land use utility. It is also possible to evaluate the effects of the construction cost or land-price on the land use transition, i.e., on our location behavior.

8 ACKNOWLEDGMENTS

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10 APPENDIX

Theorem : Given two independent probabilistic variables e_1 and e_2 that distribute following to Gumbel distribution whose parameters are $G(h_1,w)$ and $G(h_2,w)$. Then the difference e_1 and e_2, denoted by e, distribute following to logistic distribution.

Proof: Denote the Gumbel distribution function of e_1 by $F_i(e_i)$ and the joint distribution function of e_1 and e_2 by $G(e_1,e_2)$. Furthermore, denote the probability density function by $f_i(e_i)$, the joint probability density function by $g(e_1,e_2)$. Then the distribution function $F(e)$ can be rewritten as follows;

$$\begin{aligned}
F(e) &= \text{Prob}[e_1 - e_2 \le e] \\
&= \text{Prob}[e_1 \le e + e_2] \\
&= \int_{e_2=-\infty}^{\infty} \int_{e_1=-\infty}^{e+e_2} g(e_1,e_2)de_1 de_2 \\
&= \int_{-\infty}^{\infty} F_1(e+e_2)\cdot f_2(e_2)de_2 \\
&= \int_{-\infty}^{\infty} \exp[-\exp[-w(e+e_2-h_1)]]\cdot w\exp[-w(e_2-h_2)]\cdot\exp[-\exp[-(e_2-h_2)]]de_2 \\
&= \int_{-\infty}^{\infty} w\exp[-w(e_2-h_2)]\cdot\exp[-\exp[-we_2]\cdot(\exp[-we+wh_1]+\exp[wh_2])]de_2 .
\end{aligned}$$

Let $q = \exp[-we+wh_1]+\exp[wh_2]$, then we get;

$$F(e) = \int_{-\infty}^{\infty} w\exp[-w(e_2-h_2)]\cdot\exp[-q\exp[-we_2]]de_2$$

$$= \int_{-\infty}^{\infty} \frac{\exp[wh_2]}{q} \cdot wq \exp[-we_2] \cdot \exp[-q \exp[-we_2]] de_2$$

$$= \frac{\exp[wh_2]}{q} \cdot \int_{-\infty}^{\infty} w \exp[-w(e_2 - \frac{\ln q}{w})] \cdot \exp[-\exp[-w(e_2 - \frac{\ln q}{w})]] de_2 \ .$$

The integrand in the above equation can be considered as Gumbel distribution function whose parameters are $(\ln \frac{q}{w}, w)$. Hence, the value of the definite integral from $-\infty$ to $+\infty$ of the integrand makes unity. Therefore, we can reduce the above equation as;

$$F(e) = \frac{\exp[wh_2]}{q}$$

$$= \frac{\exp[wh_2]}{\exp[-we + wh_1] + \exp[wh_2]}$$

$$= \frac{1}{1 + \exp[w(h_1 - h_2 - e)]} \ .$$

Hence the theorem is proofed. If it is assume that h_1 and h_2 are the same, we will get equation (12).