Object Representation and Reasoning Using Halfspaces and Logic

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Abstract. Shapes and objects represent important ways with which to perceive and reason about the world. This paper develops a framework which uniformly encompasses both planar and non-planar surfaces to represent graphical objects in three dimensions. Based on the concept of halfspaces this framework allows the representation of volumes as predicates in logic. This representation is applied to demonstrate object concepts associated with reasoning about the topology of objects as individuals as well as groups of objects at the early, conceptual phase of designing. The example shows how both planar and non-planar boundaries of objects are treated uniformly.

1. Introduction

1.1 OBJECTS AND DESIGN

The visual world we see is inhabited by objects. Any computational model of that world needs to be able to represent objects. The standard ways of representing objects in computer-aided design systems have been largely based on the need to be able to quickly translate that representation into a screen image. This has meant that any representation has had to be numerically based. As a consequence these representations have been found to be inadequate for tasks which involve reasoning about objects in space. The effect of this has been that graphics-based computer-aided design systems have not been able to be used adequately during the conceptual design phase since it is at this phase that non-numeric reasoning is involved.

Part of the motivation for this work is the need to be able to reason about the object as well as about the relationships between objects. This is not well developed with numeric representations because of the apparent difficulties in working at the numerical rather than at some symbolic level. However, another part of the motivation to see if it is possible to develop a representation which does not have that sharp bifurcation in the modes of representations between objects bounded by planar surfaces and those bounded by non-planar surfaces as exists in most current representations. Numerically-based representations are founded on Euclidean geometry with its distinction between straight and non-straight lines and surfaces. In its modern usage this distinction can again be traced to the need to generate images quickly on computer screens. The distinction between planar and non-planar surfaces disappears when any representation is used to develop a sequence of NC manufacturing instructions. Similarly, it is claimed the distinction is inappropriate at early or conceptual stages of designing since decisions relating to planarity have not yet been taken.

There are many classes of non-numeric reasoning related to objects, some are concerned with individual objects, some with groups of objects and others with the relationships of objects with their surroundings. The first class includes semantic notions of topological relations of elements of an object such as the concept of the "upper surface". The second class includes semantic notions of topological relations amongst objects and "object emergence". The third class includes the physics of objects. The representation developed in this paper is exemplified using the first class described here. However, it has direct application in the second class as well.

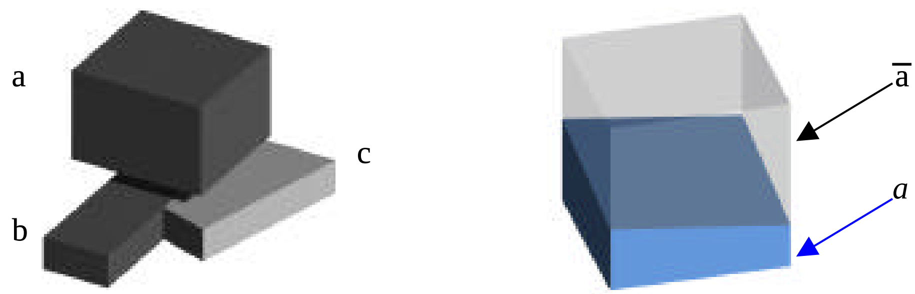
What is required here is not just one representation but two: one for numerical activity concerned with the graphical depiction of the objects as well as for numerically-based calculations and the other for reasoning about objects and their semantics. There is also a need for a direct translation from one representation to the other.

The remainder of this paper develops and describes a logic-based representation of objects capable of being used directly to reason about the topology of objects as individuals as well as groups of objects. The representation does not distinguish between objects with planar surface boundaries and those with non-planar surfaces. The second section introduces and develops a logic-based representation of objects. The third section provides the logic expressions for topological reasoning amongst groups of objects. The fourth section presents an example which utilises and demonstrates the representation.

1.2. OBJECT SEMANTICS

There is a wide variety of object semantics possible. Of particular interest here are topological and directional relations. Topological and directional relationships between objects refer to the qualitative positional relationships between objects, which often play an important role in design. Topological relations are those relations that are invariant under topological transformation.

Examples of topological relation are: intersect, disjoint, inside (and the opposite: contain), touch or adjacent to (meets at the border), equal, above, below, left, right, front and back. Some examples of such relationships can be found in Figure 1(a).



(a) (b)

Figure 1: *(a) Examples of relationships between objects:* **a** above **b**, **b** *is* adjacent *to* **c**, **b** *is* left of **c**; (b) *Halfspace* ***a*** *(represented as the shaded volume) in* **U** *and its complement,*

2. The Representation

In this paper we define a logic formalism for representing objects based on halfspaces. This formalism was introduced in Damski (1996) as halfplanes for shapes in 2D. Here we extend that representation to 3D objects and show that it also applies to halfspaces with both planar and non-planar boundaries.

2.1. BACKGROUND

A halfspace is one of two parts of a volume defined as a set of points. Each set of points defines one halfspace, as shown in Figure 1(b). In this section we present some basic definitions for the halfspace representation.

2.2. SET DESCRIPTION

Definition 1 (Universe of discourse) The universe of discourse ***U*** is a set of points *p(x,y,z)* in a reference space. There is only one universe of discourse

Definition 2 (Halfspace) A halfspace *H* is a non-empty set of points *p(x,y,z)*, $H = \{p(x,y,z) : f(x,y,z) > 0\}$ where *p(x,y,z)*   *U*. Given a halfspace *a*, there is only another halfspace $\bar{a}$ which is a set of points *p(x,y,z)* which belongs to *U* and does not belong to *a*. An example of a halfspace is the shaded space shown in the Figure 1(b).

Definition 3 (Volume) A volume V_s is the minimal set of points *p(x,y,z)* in *U* that can not be divided into a smaller set by any halfspace.

Definition 4 (Object) An object O_s is a set of points *p(x,y,z)* in *U* formed by unions and intersections of halfspaces and is the union of volumes.

Theorem 1 Every volume is formed by the intersection of all halfspaces in ***U***.
Proof: by the definition of volume.

Theorem 2 Objects are a union of volumes.
Proof: From the definition of object, it can be formed by union and intersection of halfspaces, and volumes are formed by intersection of halfspaces, therefore objects must be formed of only unions of volumes.

Definition 5 (Drawing) A drawing *D* is defined as one universe of discourse *U* which contains at least one halfspace.

Theorem 3 Given two objects O_{s1} and O_{s2} in a drawing *D*, then the intersection of O_{s1} and O_{s2} is in *D*.
Proof: All points *p(x,y,z)* in O_{s1} are in *D*, and points *p(x,y,z)* in O_{s2} are in *D*, therefore the common points between O_{s1} and O_{s2} are in *D*.

Theorem 4 Given two objects O_{s1} and O_{s2} in a drawing *D*, then the union of O_{s1} and O_{s2} is in *D*.
Proof: same as the previous theorem.

2.3. GRAPHICAL DESCRIPTION

Universe of discourse is represented graphically as a volume enclosed by a cube.

Halfspace is represented by a shaded volume bounded by a non-overlapping surface and the universe of discourse. An example of halfspace is the shaded space shown in the Figure 1(b).

Label Halfspaces are labeled by the letters *a*, *b*, *c*,....

2.4. LOGIC DESCRIPTION

Since halfspaces divide the universe of discourse into exactly two spaces, it is possible to assign a truth value True and False to each space and use it as a grounded predicate in classical logic. The mapping from a set description into logic is defined as:

Halfspace For a given halfspace *a* we define the predicate *hs(a)* with truth value δ and ¬*hs(a)* for the halfspace $\overline{a}$, where δ can be True or False. By convention, we assign the truth value True to the halfspace where its name lies. Thus *hs(a)* in Figure 1(b) is True.

Volume According to Theorem 1 a volume V_l is expressed logically by the formula: $hs(a_1)$ $hs(a_2)$ ... $hs(a_n)$, for a given *n* halfspaces in a drawing *D*.

Object According to Theorem 2 an object O_l is expressed logically by the formula V_1 V_2 ... V_m for a given *m* volumes in the drawing *D*.

Drawing Drawing is a set of Wff, well formed formulas (according to its definition in first-order logic), using the symbols *hs(x)* for halfspaces, *v(x)* for volumes and *o(x)* for objects.

This new logic representation can be used to reason about objects and their relationships.

Theorem 5 Given two objects O_{s1} and O_{s2} and their logical correspondences *o(s1)* and *o(s2)*, if O_{s1} O_{s2} then $o(s1) \rightarrow o(s2)$.
Proof: It is true that all points *p(x,y,z)* that belong to O_{s1} also belong to O_{s2}, but nothing can be said about the other points that do not belong to O_{s1} regarding O_{s2}.

Definition 6 (Reference) A reference ***R*** is a formula in the following format:
$v(x) \rightarrow hs(y)$, where a given volume is inside an halfspace.

Definition 7 (Model) A model ***M*** is a set of all possible references for a unique halfspace and its negation. A model is *minimal* $\boldsymbol{M}_{\text{min}}$ if it has the smallest number of references in a drawing.

Definition 8 (**Interpretation)** Given a volume V_l defined by the formula: $hs(a_1)$ $hs(a_2)$ ... $hs(a_n)$, an interpretation ***I*** of V_l is an assignment of truth values to $hs(a_1)$, $hs(a_1)$, ... $hs(a_n)$ and no $hs(a_i)$ is assigned both True and False.

Definition 9 (Visible) A volume V_l is said to be *visible* under a model ***M*** iff for any interpretation ***I*** in which V_l is True, ***M*** is also True. In logic, that says V_l is a logical consequence of ***M***, denoted as ***M** models* V_l.

For a given *n* halfspaces, it is possible to have 2^n different volumes. Logically, a model ***M*** can constrain this upper limit to a lower value. The definition of *visible* is important because we can distinguish what volumes can be "seen" in a particular geometrical drawing (model ***M***) from all possible volumes (upper limit).

Theorem 6 Given *n* halfspaces, there are 2^n possible volumes.
Proof: trivial.

In order to illustrate these definitions, Figure 2 shows three halfspaces *a*, *b* and *c*. The volume V_1 is described by the formula: *hs(a)*   *hs(b)*   *hs(c).*

The volumes V_1, V_5 and V_7 are examples of *visible* volumes. Since we have three halfspaces, there are 8 (i.e. 2^3) possible volumes. In this particular example only 7 volumes are visible. The volume not visible is: ¬*hs(a)*   ¬*hs(b)*   ¬*hs(c).*

An object is a combination (disjunctive normal formula) of volumes. For example, an object O_1 can be V_1 V_5, which can be expanded to: (*hs(a)*   *hs(b)*   *hs(c))*   *(*¬*hs(a)*   *hs(b)*   *hs(c))* which can be simplified further to: *hs(b)*   *hs(c)*

Finally, the *model* $\boldsymbol{M}_1$ which describes this topology is given by the formula F_1.

¬*hs(b)*   ¬*hs(c)* → ¬*hs(a)*   Formula F_1

The truth table for $\boldsymbol{M}_1$ is shown in Table 1. From Table 1 we can infer that, given *hs(x)* as non-empty halfspaces and the model $\boldsymbol{M}_1$, the visible volumes are V_1, V_2, ..., V_7. As more formulas are added to the model, the fewer the visible volumes we get. For example, in Figure 2 if we move the boundary of *c* to the right until the volume V_1 disappear there will be a need for an additional formula in $\boldsymbol{M}_1$, as shown in F_2.

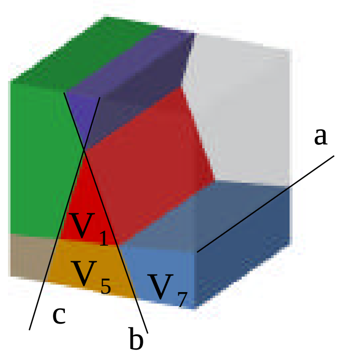


Figure 2*: Halfspaces* a, b *and* c *in* **U**.

hs(a)   hs(b) → ¬hs(c)   Formula F_2

The new model $\boldsymbol{M}_2$ is $F_1 \quad F_2$. Now the formula $hs(a) \quad hs(b) \quad \neg hs(c)$ for V_1 has the truth value False, which means it is a non-visible (empty) volume. The model $\boldsymbol{M}_2$ is:

$$(\mathrm{hs(b)} \quad \mathrm{hs(c)} \rightarrow \neg \mathrm{hs(a)}) \quad (\neg \mathrm{hs(b)} \quad \neg \mathrm{hs(c)} \rightarrow \mathrm{hs(a)}) \qquad \text{Formula } F_3$$

***Table 1**: Truth table for models* $\mathbf{M}_1$ *and* $\mathbf{M}_2$.

Volume	*hs(a)*	*hs(b)*	*hs(c)*	F_1 / $\boldsymbol{M}_1$	F_2	$\boldsymbol{M}_2$
V_1	True	True	True	True	False	False
V_2	True	True	False	True	True	True
V_3	True	False	True	True	True	True
V_4	True	False	False	True	True	True
V_5	False	True	True	True	True	True
V_6	False	True	False	True	True	True
V_7	False	False	True	True	True	True
V_8	False	False	False	False	True	False

3. Reasoning

With the representation presented in the previous section it is possible to have, at least, two types of reasoning which are of interest in CAD systems: *topological* and *directional* (relative position). In this section we present topological reasoning about objects using the halfspace representation.

3.1. TOPOLOGY

Egenhofer (1991) identifies eight fundamental relations between two shapes, six of which are of interest for objects in 3D: *touch* (meet at the border), *inside* (and the opposite: *contains*), *overlap*, *disjoint* and *equal*. Using the halfspace representation we can express the six basic relations as:

touch The volume adjacent to any visible volume V is defined as any volume V' which topologically touches V. There three ways in which two volumes can touch each other:

face Given a volume V expressed by $hs(x_1) \quad hs(x_2) \quad \ldots \quad hs(x_n)$, a volume V_{fadj} is *face adjacent* to V iff it differs in only one literal $hs(x_i)$, that is $hs(x_i)$ in V is $\neg hs(x_i)$ in V_{fadj}.

For example, in Figure 3, the volume V_2 has the description $hs(a) \quad \neg hs(b) \quad hs(c)$, therefore it has three face adjacent volumes:

$V'2_{\mathrm{fadj}}$ is $\neg hs(a) \quad \neg hs(b) \quad hs(c)$

$V''2_{\mathrm{fadj}}$ is $hs(a) \quad hs(b) \quad hs(c)$

$V'''2_{\mathrm{fadj}}$ is x$hs(a) \quad \neg hs(b) \quad \neg hs(c)$

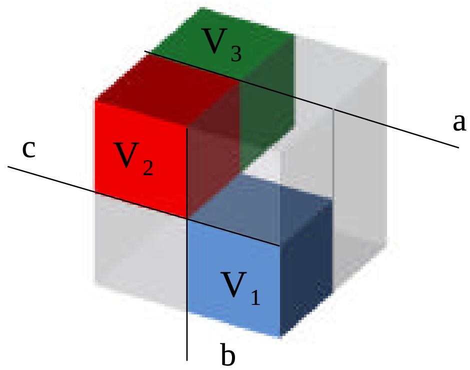


Figure 3: *Volume adjacency.*

The volume $V'2_{fadj}$ is shown in Figure 3 as V_3.

edge Given a volume V expressed by $hs(x_1)$ $hs(x_2)$ ... $hs(x_n)$. A volume V_{eadj} is *edge adjacent* to V iff it differs in exactly two literals $hs(x_i)$ and $hs(x_j)$, where $hs(x_i)$ and $hs(x_j)$ in V are $\neg hs(x_i)$ and $\neg hs(x_j)$ in V_{eadj}.

For the example in Figure 3 the volume V_2 has $hs(a)$ $\neg hs(b)$ $hs(c)$, therefore it has three edge adjacent volumes:

$V'2_{eadj}$ is $\neg hs(a)$ $\neg hs(b)$ $\neg hs(c)$

$V''2_{eadj}$ is $\neg hs(a)$ $hs(b)$ $hs(c)$

$V'''2_{eadj}$ is $hs(a)$ $hs(b)$ $\neg hs(c)$

The volume $V'''2_{eadj}$ is shown in Figure 3 as V_1.

vertex Given a volume V expressed by $hs(x_1)$ $hs(x_2)$ ... $hs(x_n)$. A volume V_{vadj} is *vertex adjacent* to V iff it differs in exactly three literals $hs(x_i)$, $hs(x_j)$ and $hs(x_k)$, where $hs(x_i)$, $hs(x_j)$ and $hs(x_k)$ in V are $\neg hs(x_i)$ $\neg hs(x_j)$ and $\neg hs(x_k)$ in V_{vadj}.

For the example in Figure 3 the volume V_1 has the description $hs(a)$ $hs(b)$ $\neg hs(c)$, therefore it has one vertex adjacent volume:

V'_{1vadj} is $\neg hs(a)$ $\neg hs(b)$ $hs(c)$

The volume V'_{1vadj} is shown as V_3.

inside Suppose two objects O_1 and O_2 are described as:

O_1 is V_1 V_2 ... V_n

O_2 is T_1 T_2 ... T_m

where Vi and Tj are volumes. O_1 is said to be *inside* O_2 (O_1 O_2) if the following expression is true:

i j | Vi = Tj

where Vi = Tj means these volumes are defined by the same halfspaces.

overlap Given the objects O_1 and O_2 described above, O_1 *overlaps* O_2 if:

i j | Vi = T_j

disjoint Given the objects O_1 and O_2 described above, O_1 is *disjoint* to O_2 if:
$i \neg \; j \mid V_i = T_j$

equal Given the objects O_1 and O_2 described above, O_1 is *equal* to O_2 iff the following expression is true:
$(\; i \quad j \mid V_i = T_j) \qquad (\; i \quad j \mid T_i = V_j)$

From these relations we can infer logical consequences. A table of such results, adapted from Egenhofer (1991), is shown in Table 2. Each cell in the table represents the possible results between shapes O_1 and O_3 from two given conditions between (O_1, O_2) and (O_2, O_3). For example, given *inside* (O_1, O_2) and *inside* (O_2, O_3) we can conclude that *inside* (O_1, O_3).

***Table 2**: Possible relations among objects O_1 and O_2, O_2 and O_3 and the implications for O_1 and O_3. The meanings of the letters are: d disjoint, t touch, e equal, i inside, ct contains, cb covered by, cv covers and o overlap.*

	disjoint (O_2,O_3)	touch (O_2,O_3)	equal (O_2,O_3)	inside (O_2,O_3)	contains (O_2,O_3)	overlap (O_2,O_3)
disjoint (O_1,O_2)	*d t e i cb ct cv o*	*d t i cb o*	*d*	*d t i cb o*	*d*	*d t i cb o*
touch (O_1,O_2)	*d t ct cv o*	*d t e cb cv o*	*m*	*i cb o*	*d*	*d t i cb o*
equal (O_1,O_2)	*d*	*m*	*e*	*i*	*ct*	*o*
inside (O_1,O_2)	*d*	*d*	*i*	*i*	*d t e i cb ct cv o*	*d t i cb o*
contains (O_1,O_2)	*d t ct cv o*	*ct cv o*	*ct*	*e i cb ct cv o*	*ct*	*ct cv o*
overlap (O_1,O_2)	*d t ct cv o*	*d t ct cv o*	*o*	*i cb o*	*d t ct cv o*	*d t e i cb ct cv o*

3.2. RELATIVE POSITION

As the mapping from a numeric to a logic representation does not carry any information about spatial relation among halfspaces, it is necessary to provide a semantic denotation for each halfspace. This denotation can be giving by assigning labels front/back, up/down, left/right for each halfspace.

Definition 10 (Vertex) The universe of discourse U is composed of eight vertices, labeled 1,2,3,...,8 as shown in Figure 4(a). Each vertex V_i is expressed by the formula: $f \quad u \quad r$ where f, u and r are three directional elements: front (opposite of back), up (opposite of down) and right (opposite of left).

Definition 11 (Denotation) Denotation is an interpretation of a vertex or the result of boolean operations among vertices.

The denotation table for all single vertices contained within a halfspace is shown in Table 3. Similar tables are required for two, three and four vertices contained within a halfspace, which can be derived from the combination of the logical formula for single ones. Five, six and seven vertices contained within a halfspace are the complement of three, two and one vertices respectively. A sample of such a table is presented in Table 4.

***Table 3**: Denotation table.*

Vertex	**Front**	**Up**	**Right**	**Denotation formulae**
1	True	True	False	$f \quad u \quad \neg r$
2	True	True	True	$f \quad u \quad r$
3	True	False	False	$f \quad \neg u \quad \neg r$
4	True	False	True	$f \quad \neg u \quad r$
5	False	True	False	$\neg f \quad u \quad \neg r$
6	False	True	True	$\neg f \quad u \quad r$
7	False	False	False	$\neg f \quad \neg u \quad \neg r$
8	False	False	True	$\neg f \quad \neg u \quad r$

***Table 4**: Some examples of denotation for more than one vertex.*

Vertices	**Example in Figure 4**	**Denotation formulae**
1,2	(d)	$f \quad u$
3,7	(e)	$\neg u \quad \neg r$
1,3,7	(f)	$(\neg u \quad \neg r) \quad (f \quad u \quad \neg r)$
4,6,8	(g)	$(\neg f \quad r) \quad (f \quad \neg u \quad r)$
1,3,5,7	(h)	$\neg r$

3,4,7,8	(i)	$\neg u$

Let the predicate *denot(a)*, the denotation of the halfspace *hs(a)*, be represented by the formula:

$denot(a)$ is $v_1 \quad v_2 \quad ... \quad v_n$

where v_i is the vertex inside the halfspace *hs(a)*. The following properties are valid for the predicate *denot(a)*:

$denot(\overline{a})$ is $\neg denot(a)$

$denot(a \quad b)$ is $denot(a) \quad denot(b)$

$denot(a \quad b)$ is $denot(a) \quad denot(b)$

The denotation of a volume is the conjunction of the denotation of each halfspace that defines that volume.

With these denotations of halfspaces, we can infer the relative position of one volume to another, in relation to a given halfspace. For example, suppose we have the two halfspaces *hs(h)* and *hs(i)* shown in Figures 4(h) and 4(i). They contain the vertices {1,3,5,7} and {3,4,7,8} respectively. The denotation of them are:

$denot(h)$ is $(f \quad u \quad \neg r) \quad (f \quad \neg u \quad \neg r) \quad (\neg f \quad u \quad \neg r) \quad (\neg f \quad \neg u \quad \neg r)$

$denot(i)$ is $(f \quad \neg u \quad \neg r) \quad (f \quad \neg u \quad r) \quad (\neg f \quad \neg u \quad \neg r) \quad (\neg \mathrm{f} \quad \neg \mathrm{u} \quad \mathrm{r})$

They can be further simplified to:

denot(h) is ¬r

denot(i) is ¬u

The denotation of intersection between *hs(h)* and *hs(i)* is:

denot(h i) is denot(h) denot(i)

denot(h i) is ¬r ¬u, or

denot(h i) is l d

So the intersection of these two halfspaces is in the *left* and *down* part of the universe of discourse.

In order to compare the relative positions between two volumes it is necessary to interpret the result of logical operations with them.

Definition 12 (Relative position) Given two volumes O_i and O_j, O_i will have relative position φ to O_j, if $O_i \quad \varphi$ is not false and $O_j \quad \varphi$ is false for all interpretations, where φ is an element in the set $f, \neg f, r, \neg r, u, \neg u$.

For example, giving the object O_1 as shown in Figure 4(b) and the object O_2 as shown in Figure 4(g) with denotations:

O_1 is $f \quad \neg u \quad \neg r$

O_2 is $(\neg f \quad \neg r) \quad (f \quad \neg u \quad r)$

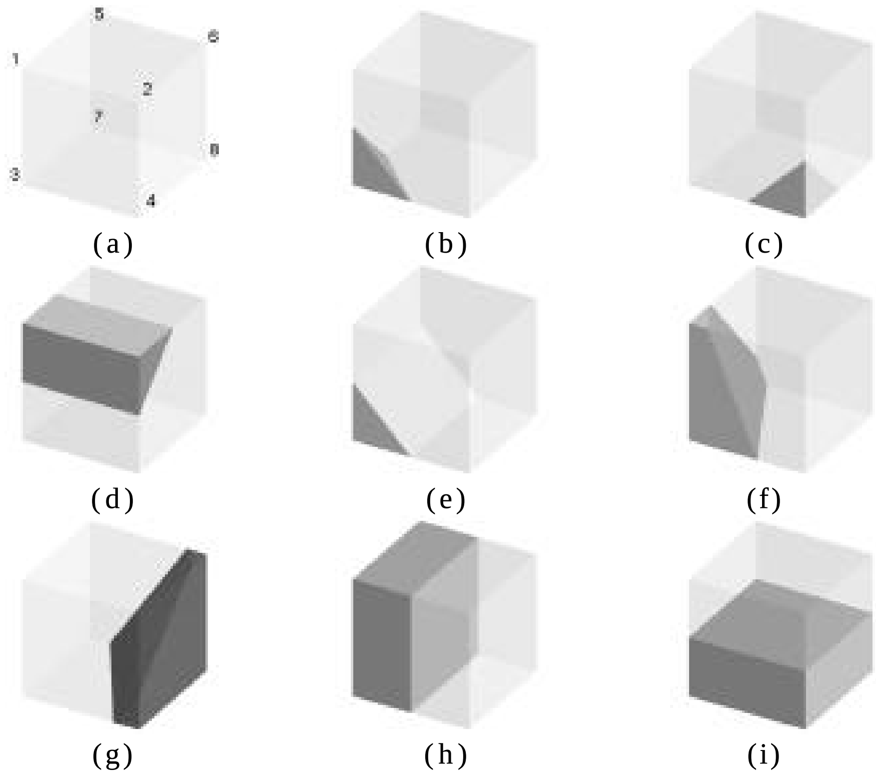


***Figure 4**: Examples of halfspace denotations.*

If we want to know if O_1 is *right* of O_2, we have to compare the formulas:

$O_1 \quad r$ which is $f \quad \neg u \quad \neg r \quad r$, and

$O_2 \quad r$ which is $((\neg f \quad r) \quad (f \quad \neg u \quad r)) \quad r$.

The first formula will be false for any interpretation and the second one is O_2 itself, therefore we cannot infer any result from this. Comparing them with $\neg r$ we have the following formulas:

$O_1 \quad \neg r$ which is $f \quad \neg u \quad \neg r \quad \neg r$, and

$O_2 \quad \neg r$ which is $((\neg f \quad r) \quad (f \quad \neg u \quad r)) \quad \neg \mathrm{r}$

In this case the first formula is O_1 itself and the second is false for any interpretation, therefore we can infer that O_1 is $\neg$r to O_2, i.e. O_1 *is not* on the right of O_2, or O_1 is on the *left* of O_2.

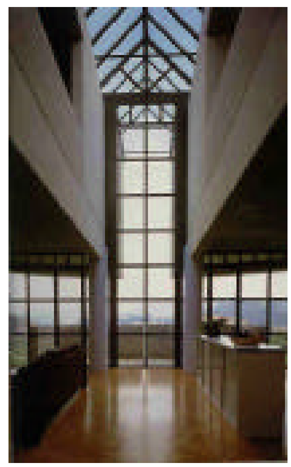
Other relations can be inferred in a similar manner.

4. Example

The representation presented in the previous sections can be used to reason about the volumes which go to make up spaces such as in a building. Figure 5(a) shows a house designed by Mario Botta (Zardini, 1984, pp. 166). This three storey house has a cylindrical body with a deep central cut from the skylight on the roof to large spaces on the first floor, where the kitchen, living room and dining room are. The first floor offers one of the most beautiful views to the outside of the house, Figure 5(b).

The architectural plans are shown in Figure 6. The ground floor contains the entrance and main hall. The first floor contains the kitchen, dining room and living room. The three bedrooms with the master bedroom and the ensuite are on the second floor.

(a)



(b)

Figure 5: *(a) Family House at Stabio, Switzerland designed by Mario Botta (Zardini, 1984, pp. 168); (b) view from the main hall and dining room (Zardini, 1984, pp. 169).*

Figure 7 shows the volumes of the bedroom, bathroom and kitchen. Let us consider the halfspaces defined within the rectangular prism shown in the Figure 8:

- *hs(a)* as the cylinder that bounds the basic structure of the house;
- *hs(b)* as the volume that limits the roof of the house;
- *hs(c)* as the volume that limits the 2nd floor;
- *hs(d)* as the volume that limits the 1st floor;
- *hs(e)* as the volume that limits the basement;
- *hs(f)* as the volume that limits the wall of the kitchen;

- *hs(g)* as the volume that limits the wall of the bathroom;
- *hs(h)* as the volume that limits the other wall of the bathroom;
- *hs(i)* as the volume that limits the wall between the bathroom and the bedroom.

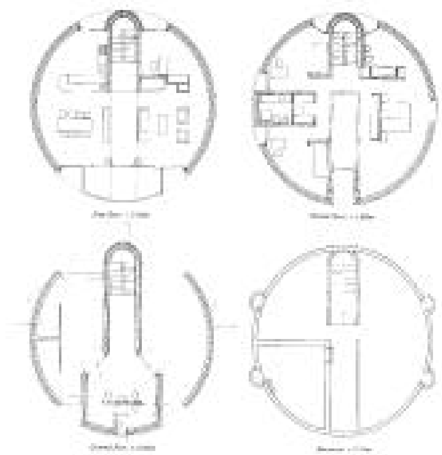


Figure 6: *Architectural plans of the house at Stabio, Switzerland designed by Mario Botta (Zardini, 1984, pp. 167).*

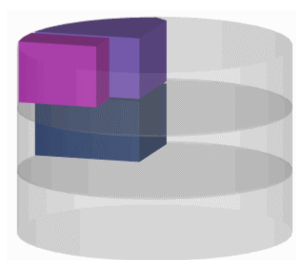


Figure 7. *The volumes of the bedroom (on top, behind), bathroom (on top, in front) and kitchen.*

The volume in the top left back corner, is one of the bedrooms and is represented by the formula:

bedroom 1 is *hs(a)* *hs(c)* *hs(f)* *hs(g)* *hs(h)* *hs(i)*.

The volume in front of bedroom 1 is the bathroom represented by the formula:

bathroom is *hs(a)* *hs(c)* *hs(f)* *hs(g)* *hs(h)* ¬*hs(i)*

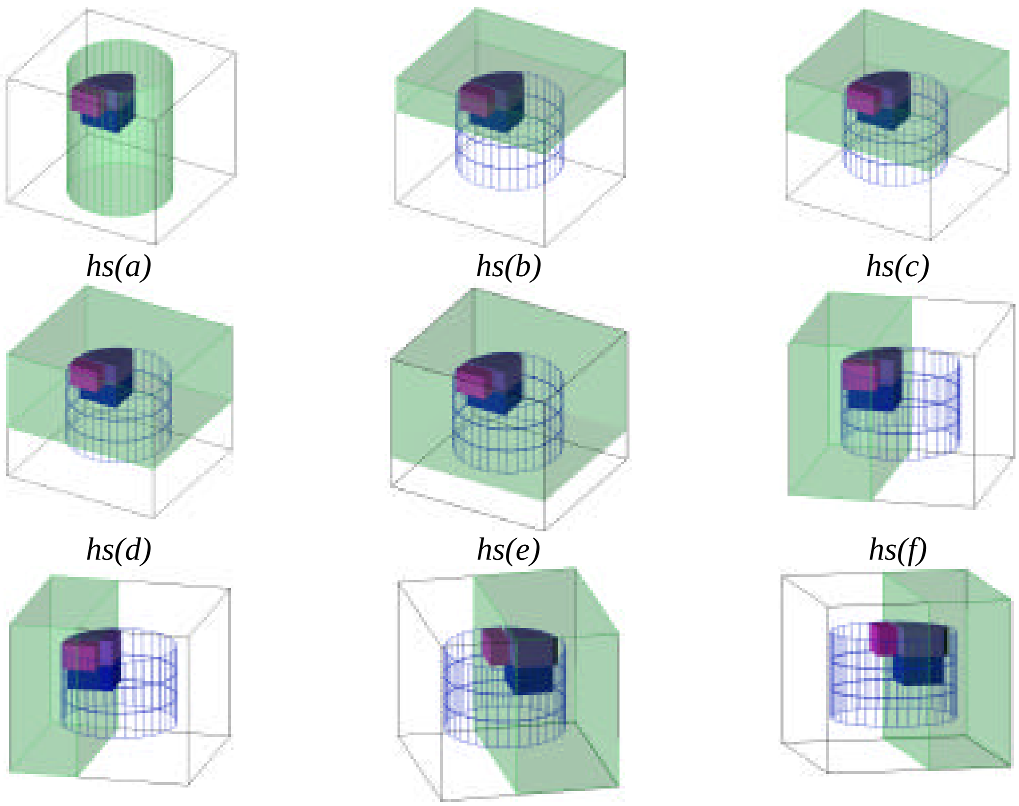
The volume underneath bedroom 1 is the kitchen represented by the formula:

kitchen is *hs(a)* ¬*hs(c)* *hs(f)* *hs(g)* *hs(h)* *hs(i)*

This example shows how it is possible to determine the relations among those volumes by inspecting their representations. For example comparing the formulas of:

Table 5: *Denotation formulas for the halfspaces in Figure 8.*

Halfspace	Vertices	Denotation formulae
hs(a)		
hs(b)	1,2,5,6	*u*
hs(c)	1,2,5,6	*u*
hs(d)	1,2,5,6	*u*
hs(e)	1,2,5,6	*u*
hs(f)	1,3,5,7	$\neg r$
hs(g)	1,3,5,7	$\neg r$
hs(h)	5,6,7,8	$\neg f$
hs(i)	5,6,7,8	$\neg f$



hs(a) *hs(b)* *hs(c)* *hs(d)* *hs(e)* *hs(f)* *hs(g)* *hs(h)* *hs(i)*

Figure 8. All the halfspaces *hs(x)* are shown as shaded volumes.

bedroom 1 and bathroom – it is possible to determine that they are adjacent to each other with the common face *i* because there is only one halfspace difference in the formula. The halfspace *hs(i)* has opposite values in the formulas.

bedroom 1 and kitchen – similar to the above condition, except that the common face is *c*.
bathroom and kitchen – the formulas have two halfspaces with opposite values: *hs(c)* and *hs(i).* This means they are adjacent to the edge of the halfspaces *c* and *i*

For directional reasoning, the cube is labelled as shown in Figure 8(a) in the same way as was the cube in Figure 4(a). Therefore it is possible to use the same table, Table 3, as the denotation table. The halfspaces in Figure 8 will have the denotation formulas as shown in Table 5.

These denotations were calculated from the disjunction of the vertex formulae. For example, for hs(b) the vertices {1,2,5,6} have the formula:

denot(b) is (*f* *u* ¬*r*) (*f* *u* *r*) (¬*f* *u* ¬*r*) (¬*f* *u* *r*)

which can be simplified to : *denot(b)* is *u*
The other combinations follow the same principle.

Bedroom 1 shown in Figure 7 as the volume in the top left back corner, is represented by the formula: *hs(a)* *hs(c)* *hs(f)* *hs(g)* *hs(h)* *hs(i).*
Its denotation formula is obtained by replacing each halfspace predicate the equivalent denotation. The replacement is: -, ¬*u*, *u*, ¬*r*, ¬*f*, ¬*f*,
The "-" indicates there is no orientation for internal halfspaces, the "¬*u*, *u*," can be eliminated because they do not represent any orientation and "¬*f*, ¬*f*" can be further simplified to "¬*f*". The resulting formula is: ¬*r*, ¬*f*, which means the bedroom 1 is on the left (¬*r*) and in the back (¬*f*) according to the orientation given in the denotation table, Table 5.

Another application of the representation is the determination of topological relations between volumes. Figure 9 shows the volume in the middle which is composed by the dining and living areas in the second floor and also the open space that connect the second floor to the third floor. This whole volume can be seen partially in the Figure 5(b). Also in Figure 9 the volume on the top right corner represents the suite (or main bedroom).

The formula for bedroom 1 in Figure 10 as shown in Figure 9 on the top right corner is composed of two volumes, G_1 and G_2 as:

G_1: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* *hs(h)* ¬*hs(i)* ¬*hs(j).*
G_2: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* ¬*hs(h)* ¬*hs(i)* ¬*hs(j).*

The final formula for bedroom 1 is: G_1 G_2

The formula for bedroom 2 in Figure 10 is:

hs(a) ¬*hs(b)* *hs(c)* *hs(f)* *hs(h)* *hs(i)* *hs(j).*

The volume for the bedroom 3, as shown in Figure 10, is:

$$hs(a) \quad \neg hs(b) \quad hs(c) \quad \neg hs(f) \quad hs(h) \quad hs(i) \quad \neg hs(j).$$

The volume for the bedroom 4, as shown in Figure 10, is:

$$hs(a) \quad \neg hs(b) \quad hs(c) \quad hs(f) \quad \neg hs(h) \quad \neg hs(i) \quad hs(j).$$

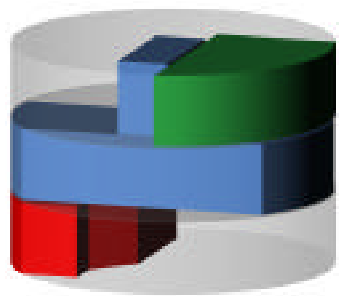


Figure 9: *The upside down tee-shaped is composed of part of the second floor and the open space in the third floor.*

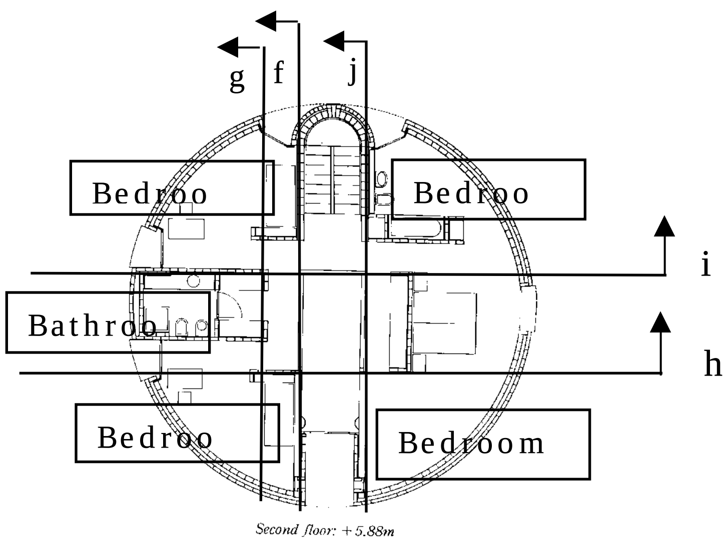


Figure 10*: Architectural plan of the third floor showing the four bedrooms, bathroom and the borders of the halfspaces g, h, i, j and f.*

The volume in the middle shown in Figure 9 is composed of five volumes, V_1 to V_5 as:

V_1: $hs(a) \quad \neg hs(b) \quad hs(c) \quad \neg hs(f) \quad hs(h) \quad \neg hs(i) \quad hs(j).$

V_2: $hs(a) \quad \neg hs(b) \quad hs(c) \quad \neg hs(f) \quad \neg hs(h) \quad \neg hs(i) \quad hs(j).$

V_3: $hs(a) \quad \neg hs(c) \quad hs(d) \quad hs(f) \quad \neg hs(i) \quad hs(j).$

V_4: $hs(a) \quad \neg hs(c) \quad hs(d) \quad \neg hs(f) \quad hs(i) \quad hs(j).$

V_5: *hs(a)* ¬*hs(c)* *hs(d)* ¬*hs(f)* ¬*hs(i)* *hs(j)*.

The final formula for this volume is: V_1 V_2 V_3 V_4 V_5

In order to find all face adjacent volumes to the volume in the middle it is necessary to apply the principles stated in section 3.1. It is necessary to find all possible volumes face adjacent to each volume that constructs that volume. For example, taking the volume V_1 defined as *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* *hs(h)* ¬*hs(i)* *hs(j)*, and fixing the halfspaces *hs(a)*, ¬*hs(b)* to keep all the volumes inside the house, there will be five possible volumes face adjacent to V_1:

A_1: *hs(a)* ¬*hs(b)* ¬*hs(c)* ¬*hs(f)* *hs(h)* ¬*hs(i)* *hs(j)*
A_2: *hs(a)* ¬*hs(b)* *hs(c)* *hs(f)* *hs(h)* ¬*hs(i)* *hs(j)*
A_3: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* ¬*hs(h)* ¬*hs(i)* *hs(j)*
A_4: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* *hs(h)* *hs(i)* *hs(j)*
A_5: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* *hs(h)* ¬*hs(i)* ¬*hs(j)*

By inspection note that the volume A_5 is the same as G_1, therefore the volume in the middle is face adjacent to the bedroom 1. In a similar way the volumes face adjacent to the volume V_2 are:

B_1: *hs(a)* ¬*hs(b)* ¬*hs(c)* ¬*hs(f)* ¬*hs(h)* ¬*hs(i)* *hs(j)*.
B_2: *hs(a)* ¬*hs(b)* *hs(c)* *hs(f)* ¬*hs(h)* ¬*hs(i)* *hs(j)*.
B_3: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* *hs(h)* ¬*hs(i)* *hs(j)*.
B_4: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* ¬*hs(h)* *hs(i)* *hs(j)*.
B_5: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* ¬*hs(h)* ¬*hs(i)* ¬*hs(j)*.

By inspection note that the volume B_2 is the same as bedroom 4, therefore the volume in the middle is face adjacent to this bedroom.

In order to find the corner adjacent volumes to the volume in the middle it is necessary to identify all volumes corner adjacent to each part of it. Volume V_1 has 10 possible volumes corner adjacent to it considering only the volumes inside the house, as:

C_1: *hs(a)* ¬*hs(b)* ¬*hs(c)* *hs(f)* *hs(h)* ¬*hs(i)* *hs(j)*
C_2: *hs(a)* ¬*hs(b)* ¬*hs(c)* ¬*hs(f)* ¬*hs(h)* ¬*hs(i)* *hs(j)*
C_3: *hs(a)* ¬*hs(b)* ¬*hs(c)* ¬*hs(f)* *hs(h)* *hs(i)* *hs(j)*
C_4: *hs(a)* ¬*hs(b)* ¬*hs(c)* ¬*hs(f)* *hs(h)* ¬*hs(i)* ¬*hs(j)*
C_5: *hs(a)* ¬*hs(b)* *hs(c)* *hs(f)* ¬*hs(h)* ¬*hs(i)* *hs(j)*
C_6: *hs(a)* ¬*hs(b)* *hs(c)* *hs(f)* *hs(h)* *hs(i)* *hs(j)*
C_7: *hs(a)* ¬*hs(b)* *hs(c)* *hs(f)* *hs(h)* ¬*hs(i)* ¬*hs(j)*
C_8: *hs(a)* ¬*hs(b)* *hs(c)* ¬*hs(f)* ¬*hs(h)* *hs(i)* *hs(j)*

C_9: $hs(a) \quad \neg hs(b) \quad hs(c) \quad \neg hs(f) \quad \neg hs(h) \quad \neg hs(i) \quad \neg hs(j)$
C_{10}: $hs(a) \quad \neg hs(b) \quad hs(c) \quad \neg hs(f) \quad hs(h) \quad hs(i) \quad \neg hs(j)$

Volume C_6 has the same formula as the bedroom 2. Also volume C_{10} has the same formula as the bedroom 3. From these conclusions the volume in the middle has, at least (other parts from this volume were not considered in this example) two other volumes that are corner adjacent: bedroom 2 and bedroom 3.

5. Conclusions

In this paper we have presented a symbolic approach based on halfspaces for the representation of objects separate from their numerically-based representation. The symbolic approach uses logic which provides a suitable processing vehicle. The obvious advantage of the logic-based representation is its ability to be used to reason about characteristics of individual objects and groups of objects uniformly, characteristics such as topological and directional relations. A less obvious advantage which has been exploited in this paper is that the representation makes no assumptions about the planarity of the bounding surfaces of objects and as a consequence it can be applied equally to objects with curved surfaces. The symbolic representation used here can be grounded in both three-dimensional with objects (as has been shown in this paper) or equally in two dimensions with shapes (Damski and Gero 1996) where all the concepts and most of the equations are directly transferable.

This separation of symbolic from the numeric representation provides opportunities for various types of reasoning in one representation not directly possible in the other. The semantics of objects and groups of objects in a context can be readily reasoned about with this symbolic representation. One form of object emergence, for example, which cannot be readily determined using the numeric representation but with this logic-based representation is directly available as the union of volumes. Path finding is available from face adjacency information, and so on. Such dual representations appear to offer many advantages.

The halfspace representation using logic can be conceived of in rather a different manner than simply an alternate representation to numerically-based approaches. It can be construed as a paradigmatic shift in representation from representing a unique individual object to representing a class of individuals of which the individual in question is an exemplar. The significance of this lies in the types of generic operations which can be carried out over the class which appears in the representation and which moves it from a quantitative representation to a qualitative one. The implication of being a qualitative reresentation is that there is the potential of using it at different stages in the development of a design, including stages which have little numerical description such as at the early stage of designing. The addition of numerical precision at a later stage does not affect the class-level representation provided the resulting individual still belongs to that class.

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