Figure 27: (left) Image of the planes perpendicular to the view direction in front of us

Figure 28: (right) Image of perpendicular to the view direction behind us

Figure 29: (left) Image of the plane which is perpendicular to the view direction and contains the origin

Figure 30: (right) Rotating an arbitrary plane containing the origin around an arbitrary axis

Figure 31: (left) Image of notable lines in perspective with six vanishing points; e: line perpendicular with the direction of view, f: line containing the origin, g: line parallel to the y-axis

Figure 32: (right) Reflection of P onto the plane containing the origin and perpendicular to the direction of view
Before beginning the discussion of perspective in western art, we should mention the contribution by al-Haytham. It was al-Haytham around 1000 A.C. who gave the first correct explanation of vision, showing that light is reflected from an object into the eye. He studied the complete science of vision, called perspectiva in medieval times, and although he did not apply his ideas to painting, the Renaissance artists later made important use of al-Haytham’s optics.

However, although Hellenistic painters could create an illusion of depth in their works, there is no evidence that they understood the precise mathematical laws which govern correct representation. First let us state the problem: how does one represent the three-dimensional world on a two-dimensional canvas? There are two aspects to the problem, namely how does one use mathematics to make realistic paintings and secondly what is the impact of the ideas for the study of geometry.

By the 13th Century Giotto was painting scenes in which he was able to create the impression of depth by using certain rules which he followed. He inclined lines above eye-level downwards as they moved away from the observer, lines below eye-level were inclined upwards as they moved away from the observer, and similarly lines to the left or right would be inclined towards the centre. Although not a precise mathematical formulation, Giotto clearly worked hard on how to represent depth in space and examining his pictures chronologically shows how his ideas developed. Some of his last works suggest that he may have come close to the correct understanding of linear perspective near the end of his life.

The person who is credited with the first correct formulation of linear perspective is Brunelleschi. He appears to have made the discovery in about 1413. He understood that there should be a single vanishing point to which all parallel lines in a plane, other than the plane of the canvas, converge. Also important was his understanding of scale, and he correctly computed the relation between the actual length of an object and its length in the picture depending on its distance behind the plane of the canvas. To give a more vivid demonstration of the accuracy of his painting, he bored a small hole in the panel with the baptistery painting at the vanishing point. A spectator was asked to look through the hole from behind the panel at a mirror which reflected the panel. In this way Brunelleschi controlled precisely
the position of the spectator so that the geometry was guaranteed to be correct. These perspective paintings by Brunelleschi have since been lost but a “Trinity” fresco by Masaccio from this same period still exists which uses Brunelleschi’s mathematical principles.

It is reasonable to think about how Brunelleschi came to understand the geometry which underlines perspective. Certainly he was trained in the principles of geometry and surveying methods and, since he had a fascination with instruments, it is reasonable to suppose that he may have used instruments to help him survey buildings. He had made drawing of the ancient buildings of Rome before he came to understand perspective and this must have played an important role.

Now although it is clear that Brunelleschi understood the mathematical rules involving the vanishing point that we have described above, he did not write down an explanation of how the rules of perspective work. The first person to do that was Alberti.

The most mathematical of all the works on perspective written by the Italian Renaissance artists in the middle of the 15th century was by Piero della Francesca. In some sense this is not surprising since as well as being one of the leading artists of the period, he was also the leading mathematician writing some fine mathematical texts. In Trattato d’abaco which he probably wrote around 1450, Piero includes material on arithmetic and algebra and a long section on geometry which was very unusual for such texts at the time. It also contains original mathematical results which again is very unusual in a book written in the style of a teaching text. Piero illustrates the text with diagrams of solid figures drawn in perspective.

He begins by establishing geometric theorems in the style of Euclid but, unlike Euclid, he also gives numerical examples to illustrate them. He then goes on to give theorems which relate to the perspective of plane figures, in examines how to draw prisms in perspective.

Piero della Francesca’s works were heavily relied on by Luca Pacioli for his own publications. The illustrations in Pacioli’s work were by Leonardo da Vinci and include some fine perspective drawings of regular solids. Now in Leonardo’s early writings we find him echoing the precise theory of perspective as set out by Alberti and Piero. He developed mathematical formulas to compute the relationship between the distance from the eye to
the object and its size on the intersecting plane, that is the canvas on which the picture will be painted.

Leonardo distinguished two different types of perspective: artificial perspective which was the way that the painter projects onto a plane which itself may be seen foreshortened by an observer viewing at an angle; and natural perspective which reproduces faithfully the relative size of objects depending on their distance. In natural perspective, Leonardo correctly claims, objects will be the same size if they lie on a circle centred on the observer. Then Leonardo looked at compound perspective where the natural perspective is combined with a perspective produced by viewing at an angle.

By 1500, however, Dürer took the development of the topic into Germany. He did so only after learning much from trips to Italy where he learned at first hand from mathematicians such as Pacioli. He published Unterweisung der Messung mit dem Zirkel und Richtscheit in 1525, the fourth book of which contains his theory of shadows and perspective. Geometrically his theory is similar to that of Piero but he made an important addition stressing the importance of light and shade in depicting correct perspective. An excellent example of this is in the geometrical shape he sketched in 1524.

Another contribution to perspective made by Dürer in his 1525 treatise was the description of a variety of mechanical aids which could be used to draw images in correct perspective.

Let us consider a number of other contributions to the study of perspective over the following 200 years. We mention first Federico Commandino who published Commentarius in planisphaerium Ptolemaei in 1558. In this work he gave an account of Ptolemy’s stereographic projection of the celestial sphere, but its importance for perspective is that he broadened the study of that topic which had up until then been concerned almost exclusively with painting.

A short time after, Wentzel Jamnitzer wrote a beautiful book on the Platonic solids in 1568 called Perspectiva corporum regularium. This is not a book designed to teach perspective drawing but, nevertheless, contains many illustrations superbly drawn in perspective.

We mentioned Commandino above and the next person who we want to note for his contribution to perspective, Guidobaldo del Monte, was a pupil of Commandino. Del Monte’s six books on perspective contain theorems which he deduces with frequent references to Euclid’s Elements. The most important result in del

Figure 2: Piero’s dodecahedron

Figure 3: Dürer’s shaded geometrical design
Monte’s treatise is that any set of parallel lines, not parallel to the plane of the picture, will converge to a vanishing point. This treatise represents a major step forward in understanding the geometry of perspective and it was a major contribution towards the development of projective geometry.

In 1636 Desargues published the short treatise La perspective which only contains 12 pages. In this treatise, which consists of a single worked example, Desargues sets out a method for constructing a perspective image without using any point lying outside the picture field. He considers the representation in the picture plane of lines which meet at a point and also of lines which are parallel to each another. In the last paragraph of the work he considered the problem of finding the perspective image of a conic section.

Three years later, in 1639, Desargues wrote his treatise on projective geometry Brouillon project d’une atteinte aux evenemens des rencontres du cone avec un plan. One can see the influence of the work from three years earlier, but Desargues himself gives no motivation for the ideas he introduced. The first part of this treatise deals with the properties of sets of straight lines meeting at a point and ranges of points lying on a straight line. In the second part, the properties of conics are investigated in terms of properties of ranges of points on straight lines. The modern term “point at infinity” appears for the first time in this treatise and pencils of lines are introduced, although that name is not used. In this treatise Desargues shows that he had completely understood the connection between conics and perspective; in fact he treats the fact that any conic can be projected into any other conic as obvious.

In 1719 Brook Taylor gives the first general treatment of vanishing points. The main theorem in Taylor’s theory of linear perspective is that the projection of a straight line not parallel to the plane of the picture passes through its intersection and its vanishing point.

Before Leonardo pondered whether the horizontal sides of a rectangle appear to be rectilinear or curvilinear, the French painter Jean Fouquet had painted transversals as curves. Can be seen the less mathematically precise versions in his work. There are a number of later examples of artists rendering straight lines as curves, but the painters found a disharmony between optics and a perspectival representation only occurred much later than the Renaissance.
In modern times we find an interesting example of curvilinear perspective that as its starting point use the same idea as Leonardo when he considered projection upon a sphere.

The twentieth-century painter, engraver, and professor of art Albert Flocon developed the idea of making a central projection upon a sphere further in collaboration with his teaching colleague André Barre. The sphere only became an intermediate stage, as Flocon generally kept to the tradition of using a flat picture plane. He and Barre therefore needed a transformation that could map an image from a sphere to a plane, and chose the so-called Postel projection. They used five vanishing points. Four vanishing points are placed around in a circle and there is one vanishing point in the center of the circle. [1] [4]

2. From axonometry to Fisheye lens

The following section we describe an outline from axonometry to perspective with six vanishing points, respecting different perspectives. Start this examination with setting an arbitrary, orthogonal coordinate system in our space, and place a rectangular cuboid. Position it so that its edges be parallel to the axes. In this coordinate system one of our eyes is in the origin while we are watching towards the x-axis. Our other eye stays closed until this paper ends. Since we would like to systematize the perspective space representations we favour being consistent rather than being expressive. Where it is needed because our arbitrarily chosen viewing-direction representation shows less expressive view – we illustrate advantages with examples.

Axonometry is one of the most popular systems which represents three dimensional objects in plane. The origins of the word are the latin axis and metrum. In every axonometric representation we have to define an orthogonal coordinate system in space, directions of the picture of x, y and z-axes and the qx, qy, qz foreshortenings. Foreshortening is the factor with which if we multiplicate the original lengths, the result is the final size of the axonometrically mapped distance. Pragmatical axonometries are not deteriorated, so none of the foreshortenings are equal with zero and non of the directions of the axes are similar. Engineeriers use that quite often, because knowing the foreshortening it is easy to find out the exact measures, but on an axonometrical figure the relative position of two points on the space is not evident. Several artists used
axonometry because of this attribute to create pictures which show impossible situations as real ones.

With perspective representation this problem can be solved. First in the row is the perspective with one vanishing points. Here we use basically two directions: normal and parallel to the plane of the figure. Parallelly with the plane of the figure we do not represent foreshortening while the lines which are normal to the plane of projection converge on one point of the horizon. This vanishing point is the intersection of the horizon and the viewing direction, the ideal point ("point at infinity") of the lines which are parallel to the viewing direction.

Perspective with one vanishing point is basically Brunelleschi’s linear perspective. With that we can represent the part of the space which can fit into a 30 to 40° cone of view.

The drawback of this system is that it represents all the objects which are at equal distance from the plane of projection but at different distance from the spectator in the same dimension. If we widen the angle of view (as first step only in one direction) this occurrence may cause disturbing distortion on the picture. So if we really need a wider angle of view we have to show render perceptible the foreshortenings, distances in all the directions where we widen the view.

Take a linear scale from -180° to 180° on the horizon. Position the 0°, the origin to the ideal point of x-axis. This scale will show us the angle, how we can see a section if one of the endpoints of the section is the ideal point of x-axis.

We can see the point a from x-axis at \( \eta \) angle, while point B is at \( \eta \). The attributes of the scale cause that the angles can be added so if we know two points and the angles which belong to them and if neither of the points fit to x-axis, we can count the angle of the section defined by the two points: line segment \( AB \) can be seen at \( \xi + \eta \).

Since we fixed the position of 0° on the scale, all the sections that fit to a plane which contains x-axis can be represented as shown on Fig. 2.5.

Now we know where the straight lines parallel to x-axis converge and also know that the lines normal to these lines have a vanishing point too. The section of these two vanishing points can be seen at 90° so we can define the exact position of this second vanishing point on our scale according to the first one.
Vanishing points are remarked ideal points, the ideal points of the lines parallel to the axes of the coordinate system predefined by us.

With the perspective with two vanishing points at the beginning of the 20th century representing objects as in reality was possible and almost perfect. But perspective representation did not stop progress. Architects created more and more precise sights, they did not change their point of view show the even bigger and taller buildings, so they had to widen the field of picture vertically. This conduced to the usage of perspective with three vanishing points. In this system we can represent an eighth part of space.

The third vanishing point can be set laterally up or down according to the current view, the rule is that developing the thought used at the one and two points perspectives the sector between the new vanishing point and the horizon must be 90°. On paper these sectors are usually as long as the sector between the two first vanishing points but that is optional.

As we extend the field of represented view, sooner or later we meet the problem: parallel lines seem to converge to different vanishing points so if we have quite a big angle of view, the picture of lines are not always linear. This contradiction can be possible because while we use perspective with one, two or three vanishing points, only half-lines, because we ignore the three other vanishing points. If two lines “meet” somewhere in front of us, it is understandable that they meet behind us too. Since every straight line has two ideal points (“point at infinity”), every axis defines two vanishing points on paper. The sector of two vanishing points defined by one axis can bee seen at 180°. The lines which are parallel to the axis of which we represented both ideal points must converge to both vanishing points.

The first perspective which can represent a whole line, so which can provide 180° field of view in one direction is the perspective with 4 vanishing points. Defining this vanishing point and all following ones means that we duplicate the representable part of space. In our example the 4th vanishing point is the second ideal point of z-axis (it is always optional in which direction we want to extend the view, naturally it depends on the object we want to represent).

Choosing between x- and y-axes to define the 5th vanishing point is also optional. Keep the ideal point of x-axis as front vanishing point and set the remaining ideal point of y-axis as 5th
vanishing point. If we extend the figure where we had expressive view, with five vanishing points we have a distorted picture (as shown on Fig. 2.12).

We can show nicer picture if we choose the vanishing points defined by z-axis as the intersection of the horizon and the sector between theese two vanishing points is exactly the vanishing point of x-axis. That is why we fixed our direction of view at the beginning of the paper. In this case our five vanishing points are in this position (Fig. 2.12).

As we duplicated the representable part of space with the new vanishing point now we can represent the half-space.

3. The sixth vanishing point

The previous section we showed that we can get the perspective we search if we initiate the remaining 6th vanishing point marked by the x-axis. But this point’s position on the paper is not so obvious and it does not follow so squarely the steps before. Many solutions of representing the whole space are known in fine arts, but these are aspire only to graphic quality for that was already the main goal of the renaissance artists.

Although it is less expressive, our method of initiating the sixth vanishing point makes us able to represent the whole space without changing the view direction. In addition with this method we can treat the whole space in a closed system in which the image of a straight line remains continuous (note the exceptions when the line go through the origin).

What do we know about the sixth vanishing point?
1. We can see the straight section between the vanishing point in front of us and behind us in 180°.
2. If the angle between one point’s position vector and the x-axis is 180° then that point is exactly behind us and its image coincides with the sixth vanishing point.

As it follows from the above mentioned conditions the 6th vanishing point’s image is a circle (as seen on Fig. 3.3 which has a radius of 180° (in our scale) and its center point is the image of the origin.
The image of a point in a perspective with six vanishing points:
With six vanishing points and mark perspective we can represent any points of which we know the x, y and z coordinates. From these coordinates we can calculate \( \alpha \) and \( \beta \) angles with the following (3.1) and (3.2) formulas

\[
a = \arctan \frac{z}{y},
\]

\[
b = \arctan \frac{\sqrt{y^2 + z^2}}{x}.
\]

In other eighth part of space we have to consider the period of trigonometric functions. We can carry out similar calculations, these are trivial.

The next step is to determine what we need to be able to describe the equation of a line in this system. Take an arbitrary set of points, for example a straight line in space. To know more about this we must get to know more about the features of this perspective.

The first step of representation is similar to the central projection of the one, two or three centers but in this case we project the points onto a sphere we sit in the center of sphere next we just project these points onto a plane.

However the final result of the representation is a plane figure, it is worth to examine the image of the shapes in the middle-phase, projected to the sphere.

Take a line in space which does not fit to the origin and represent that on the sphere around us. Denote the image as \( e' \).
We can find the exact place of the point which is the nearest from us if we also take a tangent of the sphere which fits to the plane of \( e' \). Then take a great circle which contains the x-axis and point P. The tangent of this circle at P is to be \( f' \). The angle of lines \( e'' \) and \( f'' \) corresponds with the angle of the image \( e''' \) and \( f''' \) represented in six point perspective.

Notable planes

The planes in front of us which are perpendicular to the view direction can be represented also in a perspective with five vanishing points. The \( \alpha \) and \( \beta \) value of these planes’ points can be the following

\( \alpha \in [0^\circ-360^\circ] \) while \( \beta \in [0^\circ,90^\circ] \).
The planes behind us which are perpendicular to the view direction have the same $\alpha$ value ($\alpha \in [0°-360°]$) but the $\beta$ value (where $\beta$ is the angle between the view direction and the position vector of a represented point) is $\beta \in [90°,180°]$.

In case of the plane which is perpendicular to the view direction and contains the origin $\beta = 90°$ belongs to the all $\alpha$ value.

Rotate this plan around the z-axis with a $\varphi$ angle (in our example with 45°). Now we know that this plane still contains the vanishing points marked by the z-axis. We also know the angles belong to $\alpha = 0°$ and 180°.

If $\alpha = 0°$ then $\beta' = \beta - \varphi$.
If $\alpha = 180°$ then $\beta' = \beta + \varphi$,
where $\beta'$ denotes the $\beta$ value after the transformation.

Now we rotate a plane containing the origin around an axis. The angle between our plane and y-axis is $\alpha$. We obtain the following formulas

If $\alpha = \alpha'$ then $\beta' = \beta - \varphi$,
If $\alpha = 180° + \alpha'$ then $\beta' = \beta + \varphi$.

Notable lines

Studying the lines parallel to the direction of view we noticed that every straight line has a constant $\alpha$ value and to all $\alpha$-s it assume the all $\beta$ values $[0°,180°]$. This means that the images of these straight lines remain also straight and will be a radius of the circle.

The image of those straight lines which contain the origin will be the two ideal points of the line.

Reflection

Let a point $Q$ with known $\alpha$ and $\beta$ values than reflect this point onto the plane perpendicular to the view direction and containing the origin. Thus $Q$ is moving along a line parallel to the view direction, so the $\alpha$ values of these two point are the same, and due to the features of the reflection $\beta' = 180° - \beta$.

Let’s look at the view of a cube in the middle of which we are sitting.

Knowing all the above mentioned features we can already describe geometric solids not only point by point, but also with the help of the planes and the plane surfaces bordering them.
E.g. let’s take a cube which is examined from the intersection of its diagonals, so that our view direction is normal to $\text{ABCD}$ plane. We know that $\text{ABGH}$ points are on the plane, which we get by rotation the plane normal to the view direction by taround the $z$-axis. With a similar transformation we can draw the picture of the planes defined by the $\text{BCHE}$, $\text{CDEF}$ and $\text{DAFG}$ rectangulars. The intersections of these planes define the place of the cube’s vertexes in space, and the intersections of the curves defining the images of the planes set the places of the vertexes in the plane. The $\text{AE}$, $\text{BF}$, $\text{CG}$, $\text{DH}$ edges are parallel to the view direction so we know that their images will be lines heading for the origin and also knowing the geometric features of the cube we can realize that the $\alpha$ values belonging to the four edges are $\pm45^\circ$ and $\pm135^\circ$.

Apropos of our work we recognised that besides other fields the perspective with six landmarks can be quite useful in 2D and 3D graphics. For instance imitating fisheye-effect, creating and corrigating spheric panorama-pictures and making easier and more precise to create HDRI pictures for 3D renderings.

References