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A Computational Approach for the Generation of all Partial Lattices of Two-dimensional Shapes with an n-fold Symmetry Axis

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A computational approach for the generation of all partial lattices of two-dimensional shapes with an n-fold symmetry axis is presented and an application in formal analysis in architectural design is presented in the end.

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Introduction

Group theory has been extensively used in systematic studies in analysis and synthesis of form. This could hardly be otherwise as long as group theory provides the mathematical language for symmetry and symmetry has been one of the cornerstones of formal composition in architectural design and in the arts in general (see for example, Weyl, 1952; Shubnikov and Koptsik, 1974; March and Steadman, 1974). The recent emphasis of contemporary architecture discourse on issues of pattern theory including space-packing, layer stacking, periodicity, non-periodicity, parametric variation and so on, only reaffirms the role of symmetry as a ubiquitous principle of composition in architectural design. Still, despite the usefulness of group theory in specific design contexts driven by repetition its applicability in more general design contexts can be questioned. Some first steps towards the extension of the tools of group theory to explore asymmetry – or in fact complexity, if complexity can be associated with the lack of symmetry – have been taken by March (1998), Park (2000) and Economou (2001). The hint to this inquiry lies in one of the most interesting aspects of symmetry theory and pattern analysis, namely, the part relation (<) of symmetry groups; the elaborate and complex hierarchies of symmetry groups and subgroups, all nested one within the other, point to direct correspondences with complex compositional structures of spatial patterns and suggest a precise methodology for formal analysis and composition in architecture design.

This work looks closely at a specific set of symmetry groups – the two infinite types of the planar groups, the cyclic and the dihedral ones – and provides an automated environment to enumerate and represent all their subgroups and their relationships with lattices in a graph theoretic manner. The complexity of these structures can be astonishing and it is suggested here that their graph theoretical representation can contribute to a better understanding of problems of spatial complexity in architectural design. The computational approach outlined in this work can be used in formal analysis to identify all spatial repetitions and spatial correspondences.
observed in a design. Alternatively, the approach can be used in formal synthesis to structure the design choices and bring to the foreground the whole range of spatial relationships available to the designer at any level of the design inquiry. The paper here outlines the computational approach for the generation of all the partial order lattices of two-dimensional shapes with an \( n \)-fold symmetry axis and illustrates some of these ideas with a preliminary setting of a formal analysis of the symmetry properties of the typology of courthouses.

**Aesthetic measures**

One of the most fascinating aspects of group theory is that it can provide a measure regarding the formal structure of an object; it tells the number of the parts that the object consists of and the ways these parts combine. The power of group theory is indeed its ability to uniquely define specific aspects of the structure of an object and specify the transformational structure under which these parts are related. This same power reveals its limitations though too; the perceived failure of group theory to explain objects, architectural or otherwise, that do not immediately evoke some repetitive structure has rendered this methodological tool as inappropriate for these types of formal explorations. A typical solution around this is based on ingenious usages of abstraction that plays off with vocabularies, conventions and techniques of drawing to foreground spatial elements of pictorial representation and render an architectural object with properties that in some first observation might not be observed. Often the goal is the extraction of some underlying structure that the object might be related to. A nice example is the brilliant argument by Rowe (1976) in his mathematics of the ideal villa on the alleged relationship of the Villa Malcontenta by Palladio with the Villa Stein by Le Corbusier, both abstracted in a basic symmetrical parti illustrated here in figure 1.

This work here proposes a specific way to enable these relationships and in fact suggests a method to do exactly so. The key idea that is suggested here is that asymmetry –and uniqueness and complexity and so forth– can be understood as a sum or subtraction of several parts that in themselves may have some symmetry but their products or spatial relations, render the design asymmetric. Needless to say, any superimposition or any subtraction of spaces none so ever will not immediately render interesting a space nor will say an interesting story – most probably not. What most probably will say an interesting story is the ability to reasonably discuss the possibilities that each system allows. If for example, some configuration is based, say, on a square parti,
then the transformations that are most closely related to the system are the eight ones related to the structure of the square – the four rotations of 0°, 90°, 180°, 270° and their four combinations with a mirror through the center. Now, these transformations may combine with one another to create other systems comprised by a total of 2^8 = 256 sets. Not all of them though are equal and in fact only ten of them have in some abstract way the very same characteristics of the square that specified this original system. In other words, if we insist in the example of the square parti, the compositional and organizational processes that are immediately available to the designer in this context are ten, neither nine nor eleven; whether all these will be used is irrelevant; whether they all known and are readily available is significant. The answers to all that and to the possible subsets of all known and are readily available is significant. The group has a subgroup because it satisfies all group axioms.

**Sieve**

The enumeration of all possible subgroups for a given group is a very difficult task and has been carried through only for selected few finite groups. The corresponding task of enumeration and illustration of all cyclic and dihedral groups for a given group n is straightforward but the computation is not trivial. The theory for the computation here relies on a sorting based on two theorems proved by Lagrange and Sylow respectively: Lagrange theorem identifies a very precise numerical relationship between subgroups and groups, namely that the order of a subgroup always divides the order of a group. Sylow’s theorem proposes that if a number m is a power of a prime k and divides the order of a group n, then the group has a subgroup of order m. Here the automation relies to a routine to generate the complete list of all prime factors for a given number n. The simplest, albeit not most efficient algorithm to generate the primes is the sieve of Eratosthenes. Since computing time is not a key feature for the relatively small magnitudes dealt in this project, algorithmic simplicity has been chosen over efficiency. Once the primes are extracted all possible distinct products are computed and tested to produce the possible lists of factors and the corresponding cyclic and dihedral subgroups.

The graph representation of the subgroups is a straightforward task of iterating through the factors and generating the nodes noting each time which set of operations the node represents. The completion of the illustration of the structure of the graph is done with the pictorial representation of the edges of the graph deduced from each label iterating over the nodes and arranging the nodes and the edges in a hierarchical manner for different orders of symmetry. This representation offers the most detailed
view in the inner structure of any design and illustrates nicely the stunning complexity found even in the simplest of structures. All the complete subgroup graphs of the dihedral groups up to order 12 are shown in Figure 2.

The graph representation of all symmetry subgroups of a configuration suggests a complex but rewarding insight in the symmetry structure of a spatial configuration. A more abstract view of the structure can be taken if certain characteristics of the group structure are dropped. This representation relies on a specific construct from group theory, the relationship of conjugacy or equivalence relationship and essentially produces graphs where all identical group instances are collapsed to a single node; this relation preserves the relations between the various subgroups and produces a leaner graph. Formally, given elements $x, y$ of a group $G$, $x$ is conjugate to $y$ if $g^{-1}xg = y$ for some $g \in G$. The equivalence classes are called conjugacy classes and the elements within the same class must have the same order. The conjugacy class of an element $x$ in $G$ is found by calculating $g^{-1}xg$ for every $g \in G$. Similarly, the conjugacy class of a power of $x$, say $x^n$, is found by calculating $g^{-1}x^ng$ for every $g \in G$.

All first distinct members of conjugacy groups up to order 48 are shown in Figure 3.

The computation for the conjugacy elements for each element in every symmetry group uses the Fruchterman-Reingold algorithm for the collapsed representation of the layout of the graphs (Berg et al., 2000). Essentially this computation foregrounds the qualitative difference between rotational and reflectional symmetries and classifies all subgroups and their relations in terms of two distinct classes, the cyclic and the dihedral groups respectively. The comparative illustration of all these structures reveals interesting characteristics. All subgroup lattices of a prime order have the same underlying order; familiar classes of numbers such as the square or the cube numbers are all grouped together; a rising hierarchy

Figure 2
Graph representation of the subgroup structures of the dihedral groups $D_n$ for $n \leq 12$

Figure 3
Graph representation of the conjugate subgroups of the dihedral groups $D_n$ for $n \leq 48$

a) 1; b) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47; c) 4, 9, 25; d) 6, 10, 14, 15, 21, 22, 26, 33, 34, 38, 39, 46; e) 8, 27; f) 12, 18, 20, 28, 44, 45; g) 16; h) 24, 40; i) 30, 42; j) 32; k) 36; m) 48
of networks defined in two-dimensional, three-dimensional, and four-dimensional space is nicely tabulated too. An interesting discussion on the relation of some of these numbers and particularly the highly composite Ramanujan numbers and their relation to proportional systems in architecture and specifically to R. M. Schindler’s work can be found in March (1994).

**Application**

A classical problem in formal composition in architecture is the design of central structures. The problem typically comes in several versions ranging from complete mappings of organizational structures over an overriding central parti to complex combinations of parts that emphasize locality and emergent centrality. There is an endless list of buildings and projects that illustrate these ideas and certainly there is an equally sustained effort of several architectural theoreticians, including among others Leonardo and Alberti and more recently Alexander and March, to identify the possibilities, their meaning, and the appropriate compositional machinery to resolve them.

A particularly interesting set of buildings that is related formally and functionally with the construal of center is the one of the federal courthouse buildings. These buildings, conceived and designed as sorting machines, are heavily depended on explicit sets of functional requirements and they exhibit very clear and distinct patterns of circulation that can be cast in term of point or linear spatial organizations (Dahabreh, 2006).

For the purpose of this analysis the federal courthouses are organized into four discrete zones with respect to access requirements (GSA, 1997). The public zone includes all the areas accessible to the general public along with attorneys, clients, witnesses and jurors. The private zone includes all the functions that have a restricted access and are used by particular courthouse users such as judges, jurors, and employees. The secure zone is provided for the movement and holding of defendants in custody; it includes horizontal and vertical secure circulation systems as well as holding areas. The interface zone comprises the courtroom and its associated areas; it is here that the public, private, and secure zones interact. The number, shape, size and arrangement of all the spaces within the interface zone (courtroom zone) identify many and different kinds of courtroom types. In general, these spaces in addition to the courtroom proper space, include the judges’ chambers and their support functions, jury operations spaces including jury assembly, jury deliberation and grand jury operations, security and prisoner detention areas including central holding areas and courtroom holding areas, general facility...
support areas including court administration and clerk of court, court-related agencies areas, and other building support functions. For the purposes of this work here the interface zone is considered as a point structure with the courtroom proper taken as the center and all these other functions taken as additive functional parts permuted around the core. The presence or absence of some of these spaces produces alternative functional types of interface zones. A sample of an elaborate hierarchy of types of interface zones—real and hypothetical ones—that show a cumulative addition of all the features of the interface is shown in figure 4.

The group theoretic description of the interface zone is in itself an interesting description that captures the repetitive structure of this organization. What is more interesting though is the description of the connections of these zones with one another and all with the structure of the courthouse. The key medium that does that is the secure zone because of its strict functional and spatial requirements. The secure zone is in many ways the core of the design of the courthouse because it specifies the number of the courtrooms associated with it and controls therefore the possibilities of development of the overall layout of the building itself. It is postulated here that the structure of the courthouse is specified at a great extent from the interaction of the organizational structure of the secure zone and the interface zone and more precisely from the controlled repetition of a courtroom and its associated functions. This unit is considered here as the seed for the generation of typologies of federal courthouses and it is examined in various versions and different ways of combinations of one to another. The subgroup description of the seed and its combinations provides a generous description of the complexity of the courthouse layout and in fact proposes a framework that can explain the courthouses in the corpus as well as new possible ones. A very simple illustration of the aggregation or division of the interface zone and its spatial and functional relationship with the secure zone is shown in figure 5.

**Discussion**

The major thesis that has been suggested here is that an asymmetric shape or configuration can be understood as a sum or subtraction of configurations that have discerned orders of symmetry. To that extent a computational approach for the generation of all partial lattices of two-dimensional shapes with an \( n \)-fold symmetry axis has been presented to provide an interactive catalogue with the architecture of form of all possible structures with a center of symmetry. The first steps towards an
application of this methodology in formal analysis of the courthouse typology of the US federal courthouses have been presented in the end. Current work deals with the completion of the group description of the interface zones of the courthouses in the corpus and with the construction of a full fledged group theoretical description of the complete geometry of the courthouses; this last part deals with the theoretical grounding and illustration of all formal arrangements –materialized or not– that are possible within the constraints of the existing courthouse typology.

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