

Point Worlds

Athanassios Economou¹, Thomas Grasl²

¹Georgia Institute of Technology, USA,

²Vienna Institute of Technology, Austria

¹<http://www.coa.gatech.edu/~economou>, ²<http://www.grasl.com>

¹economou@coa.gatech.edu, ²thomas@grasl.com

Abstract: *A computational approach for the automated graph representation and diagrammatic notation of all underlying symmetry structures of three-dimensional shapes with a center of symmetry is briefly presented and some applications with shape grammars to illustrate these ideas are discussed in the end.*

Keywords: *Shape studies; graph grammar; shape grammar; symmetry; configuration.*

Introduction

Group theory has been extensively used in systematic studies in analysis and synthesis of form. This could hardly be otherwise as long as group theory provides the mathematical language for symmetry and symmetry plays a significant role in formal composition in architectural design and in the arts in general. For a recent and a rather refreshing account of the mathematical language of symmetry, see Conway et al (2006) and for a generous exposition of its various applications in architectural design, see March and Steadman (1974). A particular analytical method founded upon group theory, the subsymmetry analysis, has been quite successful in showing how asymmetric designs can be seen as aggregates of parts that all possess various degrees of symmetry (March, 1998). The underlying formalism for the subsymmetry analysis, that is, the correlation of groups and graphs, has been given in Grossman and Magnus (1964). A nice series of architectural examples illustrating the method include the subsymmetry analysis of Pantheon, the Ward Willits skylight by Frank Lloyd Wright, the Free Public Library and the How house by R.M Schindler (Park, 2000) and an

extension of this method for linear designs in three-dimensional space has been given in Economou (2006). This work generalizes these approaches and provides a computational framework for the complete and automated representation of all underlying group structures of finite n -dimensional shapes with a center of symmetry, for $n \leq 3$. The application is implemented using GrGen.Net a graph rewriting system written in C#. Some possible applications to illustrate these ideas with shape grammars (Stiny, 2006) are discussed in the end.

Congruence

A central theme in formal composition in architectural design is the theory of congruence and in particular the formal relationship between a module and a whole generated by various repetitions of this module. In two-dimensional space this relationship is rather straight forward and easy to deal with and comprehend. In three-dimensional space this relationship becomes considerably more complex: the number of transformations that repeat a module increases, the complexity of the interactions of these repetitions increase exponentially, and to make

things worse – or better depending how one sees this, there is no vantage point to trace correspondences in the same way that these could be traced on a two-dimensional plane. What exacerbates the problem even more is the lack of a body of formal knowledge that could effectively help architects and designers to explore systematically the possibilities afforded in a given three-dimensional setting.

This problem of congruence in design exploration sets up a whole series of questions that are all viciously entangled one within another. It is not clear at all to begin with, whether congruence is something to be desired in design. If it is, or better, when it is, is there any preferred formalism to address such issues in a constructive way? Does this design space characterized by congruence provide a good setting for teaching formal composition (formal in both its spatial, visual sense as well as its mathematical, logical sense) in architecture? Are group theoretic computational constructs good to support such inquiry? How do shape grammars address these issues? Do structured experiments in design that are uniformed by any prior understanding of the complexities of transformations and their interactions have also research value- rather than just aesthetic value?

The work here is situated within this critical framework and sets up a computational tool to begin to address some of them. More specifically, this work looks closely at a specific set of algebraic groups – the ten abstract groups and their corresponding fourteen geometric types that can describe the symmetry of any finite 0-, 1-, 2- and 3-dimensional shape – and provides an automated environment to enumerate and represent all their subgroups and their relationships with lattices in a graph theoretic manner. The key to this inquiry lies in one of the most interesting aspects of symmetry theory and pattern analysis, namely, the part relation (\leq) of symmetry groups; the elaborate and complex hierarchies of symmetry groups and subgroups, all nested one within the other, point to direct correspondences with complex compositional structures of spatial patterns and suggest a constructive methodology

for formal analysis and composition in architecture design.

A congruence model for design

Congruence is an equivalence relation between shapes; two shapes are congruent if they can be mapped one upon the other by an isometric transformation, say a combination of translations, rotations, and reflections. An isometric transformation is a mapping that retains shape and size but changes position in space.

In architectural and spatial design congruence is typically employed to create a spatial order characterized by the repetition of similar figures. This notion of repetition links congruence to symmetry and its various construals in architecture discourse. It is a rather interesting fact that the earliest account of symmetry, at least as it presented on the first surviving treatise of architecture by Vitruvius, considers symmetry as a commensurable relation between a module and a whole (Vitruvius, 2005); our rather contemporary understanding of symmetry as a sum of congruent parts – a more precise but also somewhat impoverished term- can be traced in the theoretical exposition of architecture discourse by Alberti (1991). The Vitruvian definition foregrounds a relationship between a module and some whole cast in arithmetical terms, for example, the number of ways that a repeated module measures the whole, and the Albertian definition suggests a relationship cast in geometrical terms, for example, the kinds of ways that a repeated module can be mapped upon some whole. Nowadays, the latter relations are typically accounted in transformational geometry and the former relations in number theory and its various branches including the theory of proportion, the theory of means as well as their contemporary equivalents of the theory of modular coordination.

Congruence in formal composition in design arises and in many and diverse ways; at its simplest and most straightforward way shapes and their spatial relations can be repeated throughout the

composition to produce highly repetitive designs. Most of the designs exhibiting overt translational, reflectional, rotational, glide reflectional structure and so forth, are typical candidates for this species of the model. The next two types of congruence deal with different ways of tackling non-repetition – and complexity, if complexity can be associated with the lack of repetition. The second type of designs exhibit repetitive modules with non-repetitive spatial relations while the third type exhibit non-repetitive shapes and repetitive spatial relations. The former class consists of designs that feature identical parts put together in many and diverse ways while the third class consists of dissimilar parts that are connected one to another through repetitive identical joints. In the end of the spectrum of the model non-repetitive shapes and non-repetitive spatial relations are typically reserved for highly expressive architecture designs composed by unique shapes and components. The structure of this model can be nicely illustrated in a relation AB, whereas A denotes shape and B denotes spatial relation, and both conditioned by congruence (C) or non-congruence features (N); the four possible pairs are shown in Table 1.

Table 1
A congruence model of design

C/C	C/N
N/C	N/N

Most of contemporary designs can be cast in any of these four classes. Among these four classes of designs the first three exemplify various degrees of controlled repetition of shapes and spatial relations or alternatively of parts and joints, and they provide alternative contexts for exploration of congruence and commensurability in design. It would not be farfetched to claim that a substantial part of the fourth class of designs that rely on non-standard parts and non-standard relations can be dealt as well either under parametric rules based on given symmetry conditions or combinations of layers of

symmetry that produce unique and highly asymmetric compositions.

The work here focuses on the first design world that is characterized by congruent shapes and congruent spatial relations and explores primarily its subset of shapes that possess a centre of symmetry. In three-dimensional space the symmetries of any finite three-dimensional shape can be captured by any of the fourteen possible types of symmetry groups (Conway et al, 2006). These fourteen types of symmetry groups split into two types: seven finite types and seven infinite types. The symmetries of the seven finite classes of symmetry are described by the seven polyhedral groups that provide the symmetries of the platonic solids and their variations. The symmetries of the seven infinite classes of symmetry are described by the seven infinite types prismatic groups that provide the symmetries of any solid that has a primary axis of n -fold rotation and does not have a greater than a two-fold rotation axis perpendicular to this major axis. Both types of these geometric symmetries, the finite and the infinite types, can be succinctly described algebraically by ten abstract groups whose elements are not described in some concrete way. More specifically, the seven finite geometric types can be captured by six abstract groups, and the seven infinite geometric types can be captured by four infinite abstract groups. All these ten structures, that is, the six abstract finite groups, all closely related to the platonic solids, as well as the four infinite abstract groups, namely, the abstract cyclic group C_n of order n , the abstract dihedral group D_n of order $2n$, and their direct products with the abstract cyclic group C_2 , that is, the group $C_n C_2$ of order $2n$ and the group $D_n C_2$ of order $4n$, comprise the object of study here and provide the corpus of configurations to be systematically explored.

Sieve

Sieve is a computational tool that has been developed to address the inquiry discussed above and to enumerate and illustrate all possible subgroups of a

given n -dimensional finite shape, for $n \leq 3$. The tool has been built in successive steps each tackling a subset of the configurational possibilities of shapes with a center of symmetry. The earliest version of the software, Sieve 1.0, aimed at the group theoretic analysis of all two-dimensional designs with a center of symmetry, and produced automated graph and diagrammatic representations of all cyclic groups C_n of order n and dihedral groups D_n of order $2n$ (Economou and Grasl, 2007); the second version of the software, Sieve 2.0, aimed at the group theoretic analysis of all three-dimensional designs with a unique line of symmetry, and produced automated graph and diagrammatic representations of all four types of three-dimensional groups with an axis of symmetry n , namely, a) the cyclic groups C_n of order n , C_2 ; b) the dihedral groups D_n of order $2n$; c) the direct product groups $C_n C_2$ of order $2n$, consisting of the cyclic group C_n of order n and a cyclic group C_2 of order 2, and d) the direct product group $D_n C_2$ of order $4n$, consisting of the dihedral group D_n of order $2n$ and a cyclic group C_2 of order 2 (Economou and Grasl, 2008). The present state of the software, Sieve 3.0, concludes this inquiry with the complete automated lattice representation and pictorial representation of all symmetry subgroups of finite groups including the polyhedral groups and extends the formalism for all possible finite designs including 0-dimensional and 1-dimensional designs with a center of symmetry.

The theory for the computation for all these modules relies on a sorting based on two theorems proved by Lagrange and Sylow respectively: Lagrange theorem identifies a very precise numerical relationship between subgroups and groups, namely that the order of a subgroup always divides the order of a group. Sylow's theorem proposes that if a number m is a power of a prime k and divides the order of a group n , then the group has a subgroup of order m . Here the automation relies to a routine to generate the complete list of all prime factors for a given number n . The simplest, albeit not most efficient algorithm to generate the primes is the sieve

of Eratosthenes. Since computing time is not a key feature for the relatively small magnitudes dealt in this project, algorithmic simplicity has been chosen over efficiency. Once the primes are extracted all possible distinct products are computed and tested to produce the possible lists of factors with the corresponding cyclic and dihedral subgroups, their products and with the polyhedral groups.

The graph representation of the symmetry groups and subgroups is a straightforward task of iterating through the factors and generating the nodes noting each time which set of operations the node represents. The completion of the illustration of the structure of the graph is done with the pictorial representation of the edges of the graph deduced from each label iterating over the nodes and arranging the nodes and the edges in a hierarchical manner for different orders of symmetry.

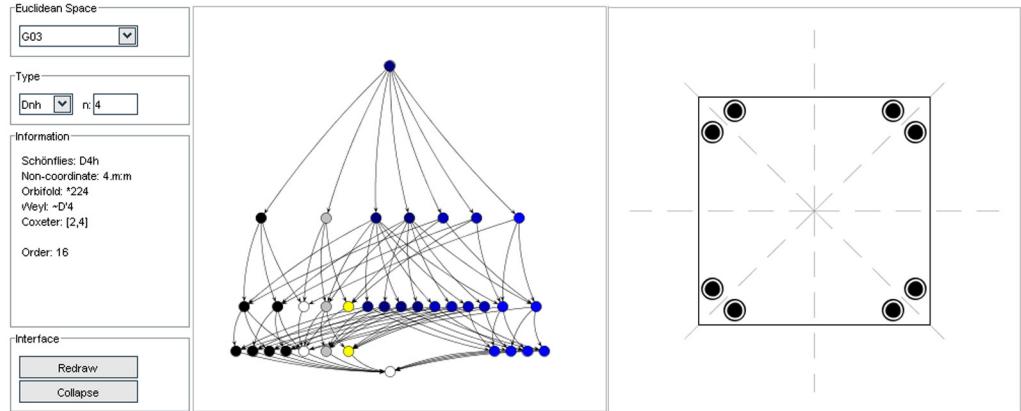
Interface

The core of the interface of the application consists of three modules that provide alternative representations of all point symmetry structures of Euclidean space. These three modules include: a) a discursive representation; b) a graph theoretical representation; and c) a diagrammatic representation in terms of labelled shapes in orthographic projections. A brief discussion of each module follows below.

The discursive module provides a set of five alternative symmetry notations for any selection of a specific symmetry structure. More specifically, this set includes the Shubnikov, Coxeter, Schönflies, Weyl and Thurston notations and they are all included here to facilitate a seamless integration of this application with major references in the field.

The graph module provides a graph representation of all nested subsymmetry groups for any selection of a specific symmetry structure. This graph representation, also known as Hasse diagram, provides a pictorial representation of the relationship of all subsets of a set; here the set of all symmetry

Figure 1
The interface of Sieve



subgroups of any symmetry group is sorted by a relation that orders all the subgroups in the set. The corresponding relations are drawn in strict order graphs if this relation can be established for all pairs of elements in the set and in partial order graphs if this relation is defined for some, but not necessarily all, pairs of elements in the set. In this notation an empty relation between elements is represented by a graph with vertices and no lines connecting them and a complete relation between elements is represented by a graph with vertices that are all connected one to another. The automated representation of the graph in the application further allows for the interactive selection for each node that corresponds to the subgroups of the structure. Each selection of a node automatically highlights the specific partial order relation of the subgroup within the total graph and updates the corresponding discursive notations in the first window of the application.

Finally, the diagrammatic module provides a diagrammatic representation of all symmetry types for any selection of a specific symmetry structure. This representation is given in terms of labelled shapes in orthographic projections, very much akin to a Wulff net projection, and arguably provides the most immediate visual output to the symmetry structure of any (finite) shape in an n -dimensional design world, and $n \leq 3$. In this representation the symmetry of the shape is given in terms of weighted

labels – the number of labels corresponds to the order of symmetry of the shape, and the type of labels – closed or open circles – signifies their relative position to the front or the back face of a shape respectively facing the viewer. It is assumed that all labels are combined with the lines of the outline of the shape to break down the symmetry of the shape.

The design of the interface of the application showing the parallel computations of discursive notations, graph lattices and diagrammatic representations of all central three-dimensional symmetry structures is given in Figure 1.

There are several nice features integrated in the design of the interface too. The nodes of the graph are all distributed horizontally and vertically to visually show the difference of the order of the nested types and their respective spatial relation to one another. All nodes of the graph can be repositioned while keeping the topology of the graph intact. All structures can be inspected at any desired degree of closeness; each time the user zooms in or out of the graph, all spatial relationships are scaled to simulate the closeness, but the nodes, the width of line, and the discursive notation of the nodes are all remaining constant to allow a common framework for illustration. Significantly, the selection of the nodes in the graph representation of the application updates the distribution of the labels in the diagrammatic representation of the

type, that is, for any selection of a symmetry subtype in a configuration the corresponding arrangement of labels show visually the nested relationship of the specific subsymmetry type within the overall type. In a rather designerly mode, the application allows as well for a composite sum of several symmetry subtypes in the configuration. In this sense the complete pattern that emerges may have no symmetry at all but its individual parts have very precise relationships that are built at will from the user of the application.

Point worlds

A first take of the integration of group lattices on formal composition with shapes and specifically within the shape grammar formalism has been given in Economou (2001). Within this framework the design is considered as an aggregation of layered simpler parts that all have some group theoretic relations within them; clearly the overall design may not have recognizable symmetries because spatial relations foregrounded by one subgroup may be obscured, subdued, transformed, or distorted by another subgroup, and the overall sum of nested symmetry structures may deny a routine understanding of a spatial composition. In that respect, the model bypasses the typical definition of symmetry as apparent repetition and with this, paraphernalia and ideologies associated with overt repetition; such understanding of symmetry is obviously within the discourse but it does not comprise the whole horizon of inquiry.

The second component of this constructive program for analysis and synthesis of designs based on symmetry properties of form requires a generative usage of the labels that are typically used to capture the symmetry structure of the design. These arrangements of labels are essentially diagrams that provide the most immediate visual output of the symmetry structure of shapes. These same labels may be used as a starting point for the design of simple shape grammars to illustrate the symmetry properties

depicted in the corresponding graph grammar. In this sense, the labels pick out the specific symmetry structure in the shape and then guide the application of rules to foreground this relationship. Within this context, the rules are of the form

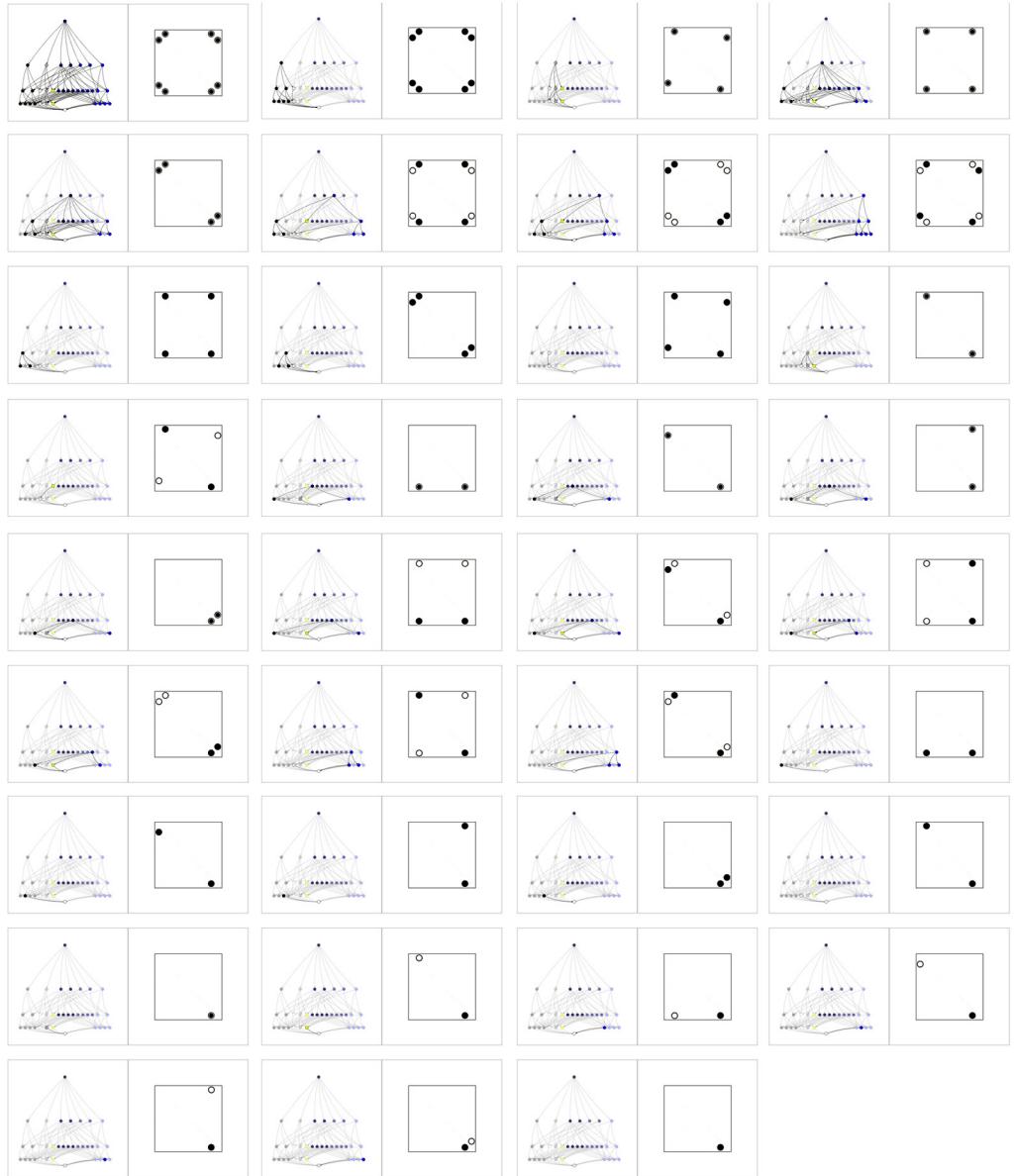
$$\bullet \rightarrow A$$

whereas the shape \bullet in the left hand side of the rule is mapped to the labels that reduce the symmetry of the shapes, and the shape A in the right hand side of the rule may be any shape in a specific spatial relation to the overall configuration. A series of simple shape grammars consisting of shape rules that use these substitution rules are readily available; still, the details of a constructive program based on these premises are left for another discussion. Instead the emphasis is given here on the framework of the model and its major thesis that such computations can produce shapes that clearly exhibit a desired degree of controlled complexity in the making of a design. The complexity of the nested symmetry structure of a shape can be staggering even for shapes with small orders of symmetry. An illustration of these ideas is given below; the shape selected for the illustration is the three-dimensional square prism. It is easily seen that the complete series of all thirty-five nested symmetrical parts within the configuration, in both graph and diagrammatic notations, requires indeed a very precise, involved, but also rewarding look in the structure of the arrangement.

Discussion

It has been suggested here that the overall emphasis of architecture discourse on issues of pattern making and parametric variation relies heavily on congruence relations. It is the purpose of this work here to examine questions regarding aspects of fitness of the role of congruence in formal composition. Currently most formal analyses and generative tools using group theoretical techniques apply or

Figure 2
An automated pictorial representation of the complete symmetry structure of a shape (square prism)



produce highly repetitive designs that show immediately their recursive structure. Still, other formalisms and especially shape grammars as well as their simple but powerful applications in kindergarten grammars, produce highly expressive designs with repeated parts that do not immediately reveal some underlying structure.

This work here reports on a computational tool that has been developed to tackle systematically this design world exploring languages and configurations (languages as sets of shapes and configurations as sets of diagrams of structure). Currently the application has been developed to enumerate and illustrate all possible subgroups of a 0-, 1-, 2- and 3-dimensional shape with a centre of symmetry. The design of this tool to help understand constructively the staggering complexity that can unfold within three-dimensional symmetry structures has been briefly described. The graph representation of all symmetry subgroups of a configuration suggests a complex but rewarding insight in the symmetry structure of a spatial configuration and the corresponding discursive and pictorial representations provide conceptual clarity and design intuition in the inquiry. The systematic import of shape grammars for design applications based on this framework is the next stage of this research project and suggests a very exciting trajectory of systematic studies in the architecture of form.

References

Alberti, L. B.: 1991, *On the Art of Building in Ten Books*, (Transl.) J. Rykwert, N. Leach, and R. Tavernor, MIT Press, Cambridge, MA.

Conway, J., Burgiel, H., and Goodman-Strauss, C.: 2008, *The Symmetries of Things*, A K Peters, Wellesley MA.

Economou, A.: 2006, *Tracing Axes of Growth*, in L. Albertazzi (ed.), *Visual Thought: The Depictive Space of Perception*, *Advances in Consciousness Research* 67, John Benjamins, Amsterdam, pp. 351-365.

Economou, A.: 2001, *Four Algebraic Structures in Design*, *Conference Proceedings of ACADIA*, pp. 192-

201.

Economou, A., and Grasl, T.: 2007, *Sieve_n*, in J. Kieferle et al, (eds) *Proceedings of ECAADE 25*, Frankfurt Am Main, Germany, pp.947-953.

Economou, A., and Grasl, T.: 2008, *Unraveling Complexity*, in J. Gero and A. Goel (eds), *Proceedings of the Third International Conference in Design Computing and Cognition*, Springer, New York, pp. 361-374.

Grossman, I., and Magnus, W.: 1964, *Groups and their Graphs*, Random House Inc., New York.

March, L.: 1998, *Architectonics of Humanism: Essays on Number in Architecture*, Academy Editions, London.

March, L., and Steadman, P.: 1974, *The Geometry of Environment*, M.I.T Press, Cambridge, MA.

Park, J.: 2000, "Subsymmetry Analysis of Architectural Design: Some Examples" *Planning and Design B: Planning and Design*, 27 (1), pp. 121-136.

Stiny, G.: 2006, *Shape: Talking About Seeing and Doing*, MIT Press, Cambridge, MA.

Vitruvius: (1914) 2005, *The Ten Books on Architecture*, (Transl.) M H. Morgan, Cambridge MA: Harvard University Press.

Weyl, H.: 1954, *Symmetry*, Princeton University Press, New Jersey.