Geometric Versatility of Abeille Vault

A Stereotomic Topological Interlocking Assembly

Irina Miodragovic Vella¹, Toni Kotnik²
¹University of Malta ²Aalto University
¹irina.miodragovic-vella@um.edu.mt ²toni.kotnik@aalto.fi

The Abeille flat vault, patented at the end of 17th century, consists of identical ashlars arranged in a woven-like pattern that generates their interlocking mutual support. In recent years, the availability of digital design and fabrication tools has caused new interest in the Abeille vault. Several studies investigate the interlocking principles through their application onto non-planar assemblies. This paper is a more systematic exploration into the underlying geometric interdependencies behind interlocking principles. It approaches the Abeille vault as a topological interlocking assembly (TIA), an assembly where basic identical elements of a special shape are arranged in such a way that the whole structure can be held together by boundary constraint, while locally the elements are kept in place by kinematic constrains imposed through the shape and mutual arrangement of the elements. The paper looks at the full potential of the Abeille vault application and studies the relation between the surface geometry and TIA

Keywords: stereotomy, Abeille vault, topological interlocking

Introduction

Stereotomy is a discipline that accumulates theoretical and practical knowledge of stone cutting and stone construction. Since, as Jacques Heyman explains: "the key to understanding of masonry is to be found in a correct understanding of geometry" (1995, 154), stereotomy is based on essential geometric relationships embedded within masonry. It started in the Gothic period, had its peak during the Enlightenment in the examples such as the undulating vaulted space of the Arles City Hall vestibule by Jules Hardouin-Mansart, completed in 1637. The main reasons for the later decline of stereotomic techniques were the introduction of steel and concrete that were less labour intensive and did not require knowledge of complex geometries, as well as change in cultural mind set where the rationalist tectonics did not approve of "deformed, showy tricks" (Evans 2000).

Stereotomic architecture is primarily based on vaulted systems to cover large spans. Such material systems ideally are shaped as an arch, as it was discovered by Robert Hooke in 1675. Hooke described the relationship between a hanging chain and the inner force flow in an arch, as "ut pendet continuum flexile sic stabit contiguum rigidum inversum" (Heyman 1998). The flat vault or voûte plate (Figure 1) invented by Josph Abeille in 1699, thus, has to be seen as an exceptional stereotomic solution that provides "all in one a ceiling for the lower storey, and a pavement for the upper storey" (Fleury 2009). In an Abeille
vault the shape of each ashlar or voussoir is identical: a polyhedron with axial sections in the shape of an isosceles trapezium. The unique ashlar geometry and the rotation of neighbouring ashlars by ninety degrees allows for a simple method of their mutual arrangement: each ashlar is carried by two neighboring ones, and at the same provides support for two other ashlars. "With this arrangement, each voussoir is carried on two others through its protruding cuts, and at the same time carries two others on its sloped cuts [...], this being reciprocal in all the vault's area, it supports itself at level" (Fleury 2009).

The woven-like organizational pattern of the Abeille vault resembles the organizational pattern of reciprocal timber frame structures of Villard de Honnecout (Lassus 1858) or Sebastiano Serlio (Hart and Hicks 1996). Both structures overcome the problem of finding a constructive solution to cover a space with a flat floor consisting of small discrete elements.

Structurally, however, the Abeille vault behaves differently than a reciprocal frame structure, namely as topological interlocking assembly (TIA), a notion introduced by Yuri Estrin and Arcady Dyskin in material science in 2001 (Glinckman 1984 and Estrin et al. 2011). TIA are defined by basic identical elements of a special shape which are arranged in such a way that the whole structure can be held together by boundary constraint, while locally the elements are kept in place by kinematic constrains imposed through the shape and mutual arrangement of the elements. TIA are scale and material independent, they are a configurational condition based on geometry. Research into TIA is currently focused on planar configuration comparable to the Abeille vault, looking primarily into configurational properties and aspects like the bearing capacity, stiffness, deformation, indentation or sound absorption (Carlesso et al. 2012, Dyskin et al. 2003a, Dyskin et al. 2012, and Dyskin et al. 2003b). The Abeille vault, thus, can be seen as a historic example for TIA, and understood not so much as a vault, but rather as network of interacting elements.

Historically, the high amount of horizontal thrust within the Abeille vault was resolved by boundary constraints like buttresses or massive walls (Brocato 2012). Due to the effort needed to support Abeille vaults it is no surprise that they were rarely used throughout history despite being "one of the most interesting technical and stylistic investigations into art of stonecutting applied to building construction" (Cities of Stone 2006).

In recent years, the availability of digital design and fabrication tools has caused new interest in stereotomy, its geometric rules, and the Abeille vault as specific case study (Etlin et al. 2008). The topological nature of the interlocking principles of the Abeille vault has been used by Giuseppe Fallacara to apply the construction principle onto non-planar vaulted structures and domes with focus on the woven aesthetics expressed in the construction principle. Oliver Tessmann (2013) has started to investigate the potential of variation within the ashlar itself and its generative effect on the shape of the assembly, as well as the potential for architectural differentiation of surface qualities. From both investigations it is apparent that the non-planar assemblies can be achieved through rotation of 'protruding' and 'sloped' ashlar faces. The ashlar configuration is directly dependant on the surface curvature, specifically the surface curvature in an underlying grid system directions.

This paper is a more systematic exploration of the full potential of the application of the Abeille vault to non-planar assemblies and studies the relation between the surface geometry and TIA. A geometric construction method is defined based on underlying...
Abeille-based TIA

The starting point of the geometrical construction method is formulation of a point grid on a given surface. For each grid field defined by vertices ABCD four points at vertices mid-span are derived: Pt0, Pt1, Pt2 and Pt3. Points Pt0 and Pt2, and Pt1 and Pt3 laying at opposite sides of the field define vectors in U and V direction respectively. Thus, each mid-point Pt that does not lie at the surface boundary defines a pair of vectors in opposite directions, Vu+ and Vu-, or Vv+ and Vv-.

A plane Pl is defined at each mid-point Pt based on the vector between the mid-point and adjacent vertex, and the sum of the vector pairs at that mid-point, referred to as Vδu or Vδv (Figure 2a). The magnitude of vectors Vδu and Vδv reflects the surface curvature in U and V directions respectively at the given mid-point. Each Pl plane is rotated around the first axes for an angle α. The rotation direction is opposite to the rotation direction of the adjacent plane (Figure 2b). The intersection of the four rotated planes defines a tetrahedron (Figure 2c).

Finally, pairs of trimming planes are defined. Each trimming plane is an offset of the plane defined by vectors Vu+ and Vv+ at the centroid Cn of each field (Figure 2d).

The Abeille-based TIA geometric construction method is, thus, based on the four main variables: the point grid distribution of the initial surface, curvature of the initial surface in U and V directions, angle of plane rotation, and the position of the trimming planes. Through establishing the interdependences between the variables the construction method is further parametrized, and the interlocking properties assessed.

First geometrical interdependency established is between the curvature of the initial surface in U and V directions, and the plane rotation angles. The values of these variables are inversely proportional: the higher the curvature, the lower is the planes rotation angle, and vice versa. In this way the elements within a planar assembly are keyed in, while elements within an area with a high curvature degree do not have awkward configurations (Figure 3).

Second geometrical interdependency established is between the curvature of the initial surface in U and V directions, and the point grid density, and is directly proportional. Through this parametrization high curvature defines high grid...
point density and smaller ashlars that in turn approximate closer the steep curvature (Figure 3).

When the initial surface is non-planar and/or point grid distribution non-rectangular, the non-adjacent elements with the same orientation intersect i.e. only in the case of the TIA of regular tetrahedra that is based on a planar surface with rectangular point grid, like the original Abeille vault, no tetrahedra intersect. The trimming planes modify and rectify elements' configuration by removing elements intersection for every other situation. The domain of possible trimming plane positions is, thus, determined by both the curvature of the initial surface and its point grid distribution.

The Abeille-based TIA geometric construction method can generate a vast number of different assembly configurations. Due to the geometric definition of the individual ashlars an Abeille-based TIA is not limited to vaulted shapes but can be defined along more general formed surfaces. From the construction sequence shown above, it is clear that the interlocking mechanism is a system based not so much on precise geometry of the intersection surface of two adjacent ashlars but rather on the orientation alternation of the intersection surface along the ashlar. It is this alternation that directs the force flow within the overall system and results in the reciprocal behaviour and the interlocking of the ashlars. Abeille vault, thus, represents a basic example of this system that conceptually lends itself for further complexification. The complexification of the system can be done by either increasing the number of polygon sides that make up the point grid, or increasing the number of alternations per ashlar face. In this way the Abeille-based TIA defines assemblies starting from TIA of tetrahedra, towards TIA of more complex Platonic solids, and ultimately free form geometries (Figure 4).

**Hexagonal Grid, Truchet Vaults and Osteomorphic Blocks**

Abeille-type configurations, are easily applied to other gridded systems, like hexagonal grids, and transformed into more fluid interlocking systems, like Truchet vaults or osteomorphic blocks. The Abeille vault’s concept of alternating plane rotations applied on a hexagonal grid generates TIA of cuboids (Figures 5a & 5b). Further, when the pairs of trimming planes are introduced the ashlar configuration becomes that of irregular octahedron (Figure 5c).

Truchet flat vault is based on the same underlying logic like the Abeille vault, with the difference that the alternating plane rotation is morphed into alternating undulation of the ashlar intersecting surface (Figure 6). The grid coincides with the vault extrados, and defines the ashlar extrados as squares, while the intrados are made up of alternating semicircular protrusion and indentations. The Truchet was developed soon after the Abeille vault as its improvement, since its interlocking ashlars leave no void on either side of the surface (Fallacara 2007). Although the solution was considered "truly ingenious" at the time it was developed the alternation of concave and convex surfaces was found to difficult to actually execute (Etlin et al. 2008) until the introduction

Figure 3
Curvature, point grid distribution and plane rotation interdependency.

Figure 4
Diagram of Abeille-based TIA complexification.
of CAD/CAM processes.

The osteomorphic block is based on the same concept of Truchet's alternating concave and convex surfaces, with the difference that the alternation is developed along the same ashlar face (Figure 7). The interlocking is achieved only along one axes and since the system was invented for a wall and column construction, osteomorphic block is not based on a grid, but a running bond (Dyskin et al. 2003c).

Further complexifications investigated in the introduction of perforations within the Abeille-type system. Perforations can be achieved by manipulating the tetrahedral ashlar configuration. Tessman subtracts volumes form the tetrahedral ashlar to investigate the variations of the intersecting surface contact zones. The resulting cropped ashlars define perforations while maintaining the interlocking principle (Tessman 2013). Perforations can be achieved also through variation of the trimming plane positions, specifically when both trimming planes are placed either below or above the initial grid (Figure 8). To ensure the interlocking within the system, further parametrization is established between the ashlar thickness and distance from the grid between neighbouring pairs trimming planes.

Looking at the examples shown, it can be concluded that for defining the interlocking ashar geometry, it is enough to consider the alternation only along the parameter of the grid field i.e. an 'inward' rotation has its identical, corresponding 'outward' rotation on the other side of the same ashar. This is the same principle used in two dimensions by Escher in a mastery way in his planar tessellations (Figure 9) (Haak 1976).
Conclusion

The discussion shows that the Abeille-like configuration can be seen as generic example of TIA. Other examples like the Truchet vault or osteomorphic blocks and their parametric variation can be understood as simple variation of the principles that define the Abeille-vault. Such a unifying approach allows a more systematic study of the structural behaviour of TIA and the limitation of the material system with respect to the geometry of the underlying surface. TIA are characterized by a systemic transfer of loads based on the interaction of neighbouring ashlars. This way a cyclic distribution of loads appears within the material system that resembles the non-hierarchical load pattern in reciprocal frame structures (Kohlhammer et al. 2011). A more detailed analysis is required, however, to understand in more detail the influence of curvature on the structural behaviour of TIA.

The local interaction of the ashlars is not only providing a more even distribution of loads within the structure itself but at the same time providing a local stabilization of the configuration that has positive effects for the assembly. First tests that have been conducted with an Abeille-like pavilion structure show that the interlocking of the ashlars enables a significant reduction of falsework that is typically needed for the construction of masonry vaults. This means that TIA have the potential to increase again the interest in vaulting structures that is in a building typology that had not been used very much in developed countries since the late 1970s due to the high amount of labour that is needed for the preparation of falsework.

This study on Abeille-like configurations therefore demonstrates clearly how the combination of digital design tools and digital fabrication enables the reanimation of historic construction techniques for the benefit of contemporary architecture. In addition, the geometric nature of the Abeille-like construction allows for a more holistic understanding of the interplay of geometric relationship within the assembly, specifically surface curvature and the config-
uration of individual elements, and enables the transfer of structural principles between different construction typologies which in turn helps to delimit system boundaries and opens up the possibility of a more fluid understanding of structures in architecture.

REFERENCES
Dyskin, AV, Pasternak, E and Estrin, Y 2012, 'Mortarless structures based on topological interlocking', Frontiers of Structural and Civil Engineering, pp. 1-10
Fallacara, G 2007, Towards a Stereotomic Design, Aracne editrice S.r.l., Roma
Fallacara, G 2009 'Toward a Stereotomic Design: Experimental Construction and Didactic Experiences', no source given
Fleury, F 2009 'Evaluation of the Perpendicular Flat Vault Inventor’s Intuitions through. Large Scale Instrumented Testing', Proceedings of the Third International Congress on Construction History, Cottbus
Gallon, J. 1735-1777, Machines et inventions approuvées par l’Académie Royale des Sciences depuis son établissement jusqu’à présent, Académie des Sciences, Paris
Glickman, M 1984 'The G-block system of vertically interlocking paving', Proceedings of the 2nd International Conference on Concrete Block Paving, Delft, Netherlands, April, pp. 10-12
Guerrieri, CD (eds) 2006, Cities of Stone, Fondazione La Biennale, venezia
Hart, V, Hicks, P and Serlio, S 1996, Sebastiano Serlio on architecture, Yale University Press
Heyman, J 1998, Structural analysis: a historical approach, Cambridge University Press
Lassus, JBA and Darcel, A 1858, Album de Villard de Honnecourt, architecte du XIIIe siècle: manuscrit publié en fac-simile, annoté, précédé de considérations sur la renaissance de l’art français au XIXe siècle et suivi d’un glossaire, Imprimerie impériale
Tessmann, O 2013, Interlocking Manifold Kinematically Constrained Multi-material Systems, Springer