Square tessellation patterns on curved surfaces:

In search of a parametric design method

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Methods for Tessellating a flat surface with regular or semi-regular patterns of polygons have already been addressed in literature and can be easily parameterized. For the tessellation of curved surfaces using patterns of one or more regular polygons there is not a uniquely defined approach to the problem within the context of architectural research and applications. This paper is focused on the tessellation of curved surfaces with square tiles, where the tessellation pattern consists of four squares with partly overlapping sides. In this study double curvature surfaces were considered first, and subsequently surfaces of more complex geometry such as minimal surfaces. Specifically, a method for the square tessellation of two types of doubly curved surfaces, the spherical and the ellipsoidal, is discussed and presented in the paper. In addition, the square tessellation of two types of minimal surfaces, the catenoid and the helicoid, have also been examined and presented. For each one of the surfaces that have been considered, an algorithm that generates the distribution of the planar square surfaces on the surface and renders possible the parametric description of the problem, was developed and presented in the paper. A discussion on boundary conditions for each developed method is also included. The Grasshopper visual programming language has been used for the parametric description and display of the results in a graphic environment. The research discussed in this paper can find application in several real world problems including surface paneling, or space packing of polyhedral structural units on a curved surface.

Keywords: *square tessellation, curved surface tiling, ellipsoid tessellation, minimal surfaces tessellation, geometric appxoximation methods*

INTRODUCTION AND BACKGROUND

The geometric concept of surface tiling or tessellation, defined in simple terms as the covering of a surface by surface units of one or more shapes in such a way that there are no spaces between, and no overlapping of the units, has found many applications in historical and contemporary architecture. Examples of curved surface tessellations in building design include tilings of domical surfaces of spherical or cylindrical shape in interior or exterior building surfaces.

Tessellating flat surfaces with regular or semi-regular patterns constitute geometric problems with known solutions, i.e. there are eight semi-regular tilings of the plane. The tilings in this case are described by their vertex configurations and can be easily parametrized. On the other hand, tessellations of curved surfaces using a single regular polygon, or combinations of two or more regular polygons, require investigation as there are not always uniquely defined solutions within the context of architectural research and applications. Indeed, in some cases, the question of curved surface tessellation, that involves the application of a pattern on a surface, can be considered as a special instance of surface subdivision problem that has been addressed in both regular and free form geometry literature including mesh subdivision processes (Jiang, M., & Wonka, P. 2014). In the same context of free form geometry, curved surface tessellation problems can be addressed as a special case of surface paneling also covered in recent literature (Bommes, D., et. al. 2013). However, when very specific constrains apply with regard to the geometric characteristics of the composing shapes in a pattern, such as planarity, dimensions, etc., then determining a solution may constitute a very difficult problem or simply one with no solution.

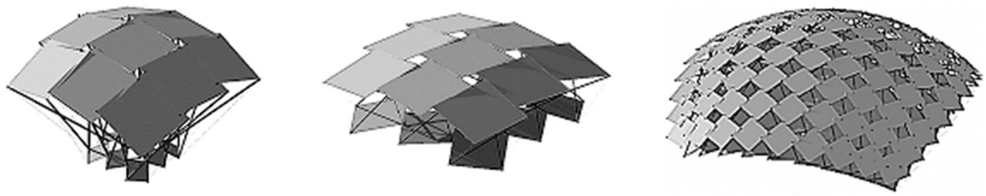
The research in this paper addresses the application of a geometric pattern on curved surfaces that which can be described using mathematical equations. Although this problem, at a first glance, may appear as an easier case than the application of a pattern on a free form surface, it also involves a great deal of complexity and in most cases requires geometric approximation approaches. Within this general frame of surface tessellation questions, the paper is focused on the tessellation of curved surfaces with planar square tiles, where the tessellation pattern consists of four squares with partly overlapping sides. In this study curved surfaces of regular geometry are considered first, and subsequently surfaces of more complex geometry such as minimal surfaces.

SQUARE TESSELLATION PATTERNS ON A SPHERICAL SURFACE: GEOMETRIC CONSIDERATIONS AND SOLUTION

In architectural practice, there are several instances where there is a need for determining a method for covering a curved surface with planar surface elements. The covering of a surface of regular geometry with planar panels, where the panels need to maintain a square shape and a fixed connection pattern throughout the surface is one of them. Environmental applications, involving the integration of solar panels on a curved roof could be another example. Furthermore a tessellation problem with square shaped panels that form a pattern may refer to the covering of a reciprocal structure frame.

The space packing of polyhedral structural units on a curved surface constitutes another field for potential application of the question at hand. As a specific instance of space packing problems, the design and construction of tensegrity double layer structures composed of units of square base also constitutes another field for potential application of this geometry. In this case, maintaining the square tessellation pattern on both the layers of the structure is a requirement. Departing from this specific research topic, earlier research has focused on the development of methods for generating a four square pattern on a spherical surface (Liapi, K. 2001; Liapi, K. and Kim, J. 2005). In this case the four square pattern results from the connection of the tensegrity units on each layer of the structure. In the method already developed by Liapi an effort was made to restrict the solution to the problem to only one size of square -base units throughout the structure.

Figure 1
The appropriateness of the resulting geometry, based on constructability criteria, greatly depends on the ratio between the radius of the spherical surface and the length of the side of the square unit.



As shown in Figure 1, in the above mentioned method that involves the parametric generation of a double layer tensegrity surface of spherical or domical geometry, the appropriateness of the resulting geometry, based on constructability criteria, depends on the ratio between the radius of the spherical surface and the length of the side of the square unit. By decreasing the value of the ratio, the accuracy of the tessellating pattern also decreases. This consists a limitation of the method.

In the current study, the constrain according to which only one size of identical square-based units should be used on the entire surface has been waived, and the progressive reduction of the size of the two composing square- based units of the pattern has been allowed. In brief the objective of this study was to come up with a parametric method that: a) will not depend on the size of the radius b) will allow for the uniform distribution of the squares throughout the surface of the sphere.

Working In this direction, we approached the problem of the spherical surface tessellation by following various existing methods. A literature review of the Mercator projection method, also named isocylindrical projection was considered early on in this research. A main feature of this method is that it can generate a network of points which, when connected, form lines that are perpendicular to each other (parallels and the meridians) (Deakin 2002; Karney 2011; Osborne 2013).

In order to simplify the problem at hand the four squares of the pattern are replaced by a point at their centers. This reduces the tessellation pattern to a square tiling problem. To solve this tiling problem the distances between the points need to be properly determined, so that, once replaced by squares, or near-squares, their sides should be touching each other and the connection pattern should be maintained. To achieve this, a code that constructs a sphere by applying its parametric equation, that is x=acosucosv, y=bsinucosv και z=csinv where the fields for u και v are defined respectively as $0<u<2\pi$ and $-\pi/2<v<\pi/2$ was developed first.

Then, a network of equidistant points on the parallels and the meridians was constructed. Since the distances of the points along each parallel is fixed, while, along the meridians the distances of these points towards the poles are increasing, when connecting these points to form squares, the squares become rectangles as they come closer to the poles (Figure 2a). This specific problem has already been addressed in cartography by applying the Mercator projections which in essence can increase the number of parallels while moving towards the poles. So at a following stage, a code that redistributes these points by inverting the parametric equations of the Mercator projection of the sphere $x=R\lambda$ and $y(\varphi)=R\psi(\varphi)=R\ln(\tan(\pi/4+\varphi/4))$ was developed (Figure 2b). The new distribution of the points rendered possible the creation of adjacent near-squares (Figure 3c).

For the creation of the four square tessellation pattern on the sphere, the centers of the squares in Figure 2c, were placed first. So the code included a function that places the centers of gravity of all the squares on the surface. Once all the points were translated in lists and the proper descriptions were made, a network of squares was generated.

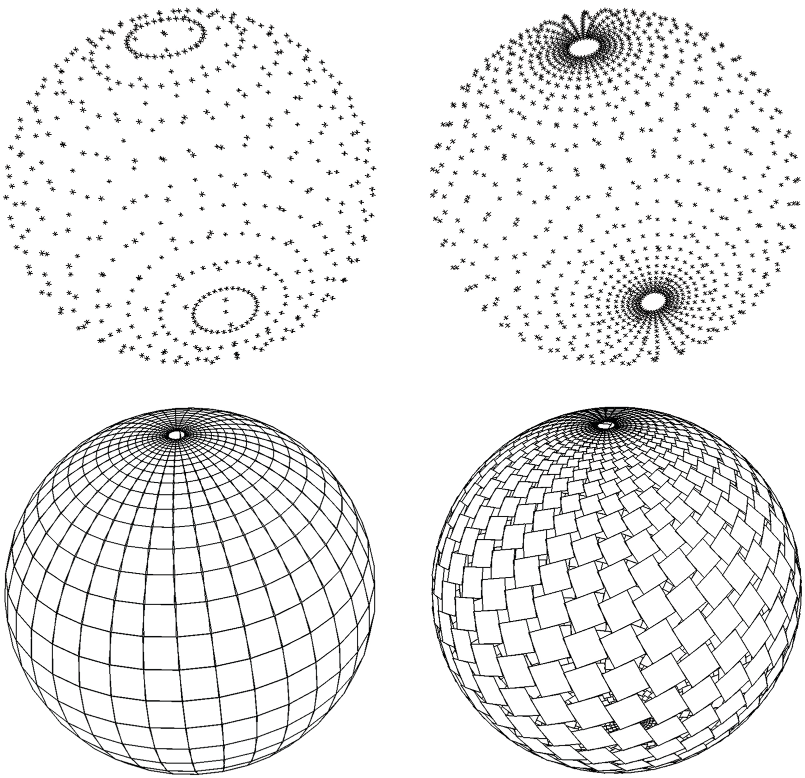


Figure 2
a) Point distribution on the sphere with the use of parametric equations, b) Point distribution on the sphere after the application of the Mercator projection, c) Creation of adjacent shapes near-square shapes on the surface of the sphere and d) Rotation of the square units around a perpendicular axis that passes through their centers to form the four-square tessellation pattern on the sphere.

In order to utilize the developed network of squares on the surface of the sphere for the creation of the four square pattern, some additional geometric rules were applied. Specifically the squares that represents the 'void' between each set of four adjacent squares were created by rotating each one of the squares on the surface of the sphere by the same angle around a perpendicular axis that passes through its center. The rotation angle depends on the desired overlap of the sides of the adjacent squares that form the pattern. At the same time, the code calculates the required change in the size of the squares (scale) after rotation so that their sides still overlap (Figure 2d). It is interesting to note that, as the rotation angle increases, the size of the squares that compose the pattern decreases and respectively the void space between adjacent squares increases. A value that constrains the rotation angle so that the overlap between adjacent squares is maintained is also determined by the code.

GEOMETRIC SOLUTION OF THE ARRANGEMENT OF A FOUR SQUARE PATTERN ON AN ELLIPSOID

Departing from the problem solved previously that refers to the geometric solution of the arrangement of a four square pattern on a sphere, the creation of a set of points on the surface of an ellipsoid has been attempted. In fact an ellipsoid with the same set of points as the sphere is created by changing the radius of the sphere on the z axis. The Mercator projection technique, that is, the projection of the points on the surface of the sphere to a plane, provides a 2D network of points that are equidistant and form square panels on the 2D plane. However, by repeating the next step of the projection process for the ellipsoid, we can easily observe that the points on the y axis are no longer equidistant and, as a result, the projection of the points on the surface of the ellipsoid does not form square panels. Therefore, a method that will allow us to redistribute these points on its surface was needed.

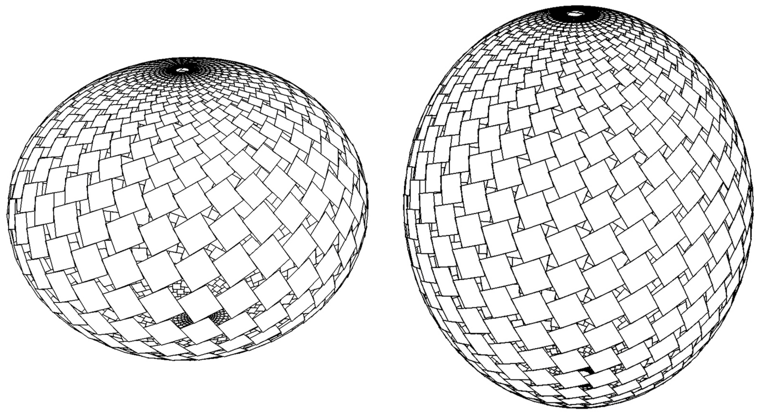
Various techniques have been examined in order to proceed with the appropriate geometric transformations. A method that approaches the problem graphically, using numerical approximations is also developed. Specifically, in order to approximate the point position change rate along the axes, sinusoidal curves are created to represent this proportion, which are translated afterwards to polynomial expressions.

Specifically, the discrete values that indicate the ratio of the z axis to the x or y axis of the ellipsoid are taken (such as c = 1.1, c = 1.2...c = 2.0) and the curves that represents the rate of the change of the points in each case were derived. These curves are sinusoidal and their rate of change remains fixed as the dimensions of the z axis increased. Based on this, an equation that generates the mapping of the points of these curves on the surface of an ellipsoid was developed. By changing these curves in a MATLAB environment to 7th order polynomial expressions and by inserting the coordinates of various points on the curves, a set of points that reflect the square patterns on the surface of ellipsoids of various ratios (c = 1.1, c = 1.2 ... c = 2.0) were generated as shown in Figure 6. As for example for the case of c=2.0 the polynomial expression av + bvˆ3 + cvˆ5 + dvˆ7 was used, where a=0.003, b=-0.015, c=0.180 and d=0.5015.

This followed process made clear that in the case of the ellipsoid a purely analytical expression to generate the four square pattern on the surface of its surface could not be determined. Instead, for the solution to the problem an approximation method had to be used. It was also shown that by decreasing the size of the squares, the square pattern on the surface of the ellipsoid is not affected (Figure 4).

Subsequently, other approximation methods for tessellating the surface of an ellipsoid were explored.

Figure 3
Square tessellation on ellipsoidal surfaces of various ratios between the two main semi-axes of the ellipsoid. The change of the ratio does not affect the accuracy of the method.



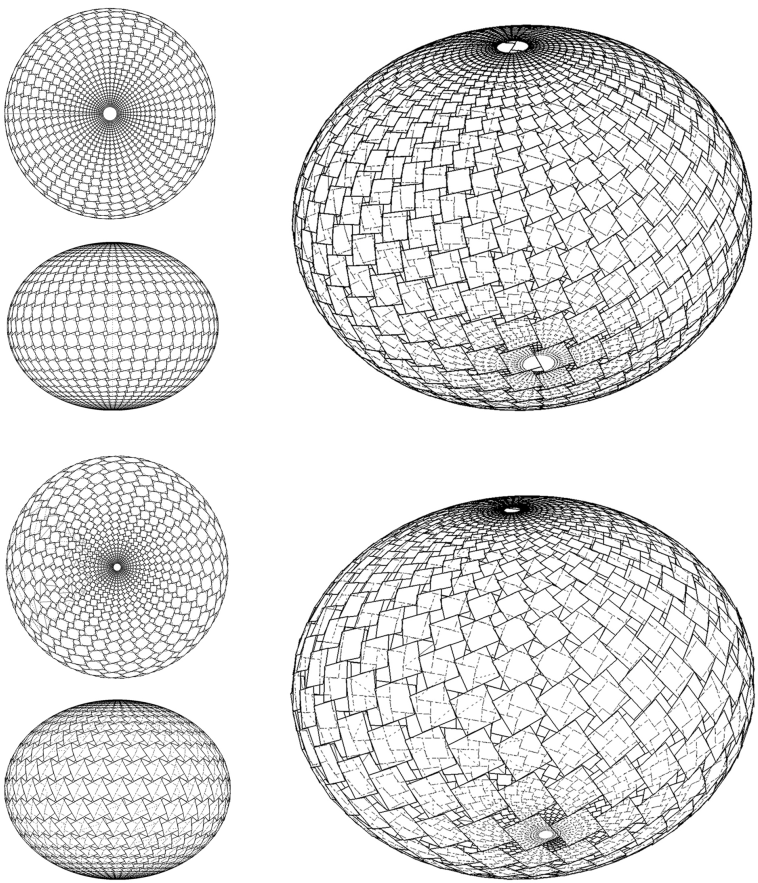


Figure 4
By increasing the size of the squares the square pattern on the surface of the ellipsoid is not affected

So, in addition to the polynomial approximation, a second method that makes use of fixed point iterations (Osborne 2013) was also explored. It is shown that the accuracy of the method is not affected when the ratio between the two main semi-axes of the ellipsoid changes (Figure 3).

Both methods generate highly accurate results throughout the entire surface; their accuracy depends on the number of the polynomial terms and the number of iterations respectively.

GEOMETRIC SOLUTION OF THE ARRANGEMENT OF A FOUR SQUARE PATTERN ON MINIMAL SURFACES

Minimal surfaces are defined as surfaces with zero mean curvature and may also be characterized as surfaces of minimal surface area for given boundary conditions (1). Despite the complexity that most of the times characterizes their form, minimal surfaces, because of their specific geometric properties, can be tessellated with planar square patterns by applying

Figure 5
Square tessellation pattern on a catenoid before and after the rotation

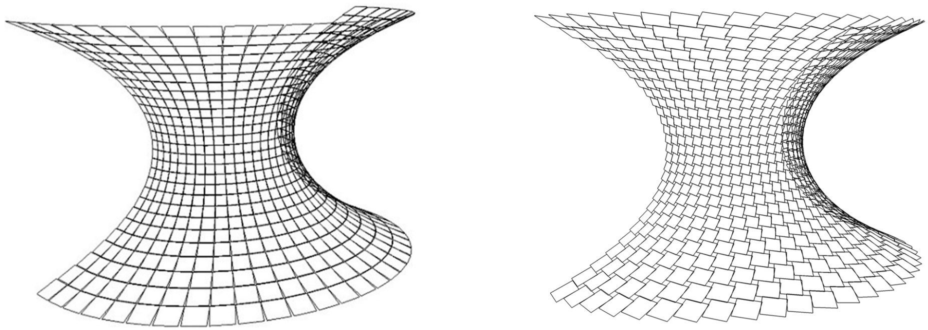
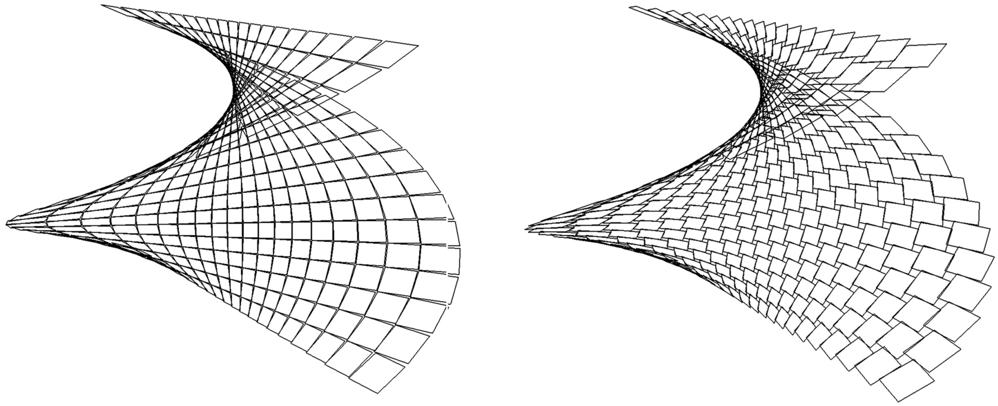


Figure 6
Square tessellation pattern on a helicoid before and after the rotation



the method describing earlier without the need for any modifications. Accordingly, for the generation of a network of equidistant points on α minimal surface, and subsequently for the tessellation of the square tile pattern, the parametric equations of the surface suffice and there is no need for any approximation method. Following this method two types of minimal surfaces, the catenoid and the helicoid, have been examined as shown in Figures 5 and 6.

For each one of the used methods, an algorithm that generates the distribution of the planar square surfaces on the curved surfaces and renders possible the parametric description of the problem, was developed. For the iterations, a code, written in Python, was inserted in the basic code. The Grasshopper visual programming language has been used for the parametric description and the display of the results in a graphic environment. The visualization of the results has facilitated the comparison of the developed methods.

5. CONCLUSIONS

This study successfully addressed the tessellation problem of curved surfaces with a four square pattern.

At the first stage of this study, the distribution of a pattern of square tiles on the spherical surface was studied and an algorithm has been developed. A code that utilizes the Mercator projection for the implementation of the algorithm was also developed.

At a following stage, a parametric method that allows the uniform arrangement of a four square tessellation pattern on the surface of the ellipsoid has been developed as well as a code that implements the steps of the algorithm. The developed method involves geometric approximation processes. The Grasshopper visual programming language has been used to this end and the boundary conditions for the arrangement of the square pattern on both the surface of the sphere and the ellipsoid have been determined.

The visualization of the results in the graphical interface of the grasshopper programming environment has facilitated the study of the effect of various parameters of the problem and the comparison of the results of the developed method.

The developed algorithms and parametric processes are expected to offer a valuable design tool that will facilitate the exploration of the application of planar square tessellation pattern on a wide range of curved surfaces, including spherical and ellipsoidal, as well as certain types of minimal surfaces.

ACKNOWLEDGEMENTS

The current study is part of a research project supported by the "IKY Fellowships of Excellence for Postgraduate Studies in Greece - Siemens Program" and by the Grant E.037 from the Research Committee of the University of Patras (Program K. Karatheodori).

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