Geometry, Light, and Cosmology in the Church of Hagia Sophia

Wassim Jabi and Iakovos Potamianos

Designed by a physicist and a mathematician, the Hagia Sophia church in Istanbul, Turkey acted as an experimental test case in which advanced knowledge of geometrical constructs, sophisticated understanding of light behavior, and religious and cosmological beliefs combined to create a magnificent structure. While some of its design concepts are known, many remain hidden. Earthquakes have demolished parts of the church—such as the original dome. Researchers have in the past misinterpreted their observations and perpetuated false conclusions. Lastly, the lack of digital tools has until now prevented verification and analysis of prior findings. In this paper, we integrate traditional historical research, parametric digital analysis, and lighting simulation to analyze several aspects of the church. In particular, we focus on the geometry of the floor plan, the geometry of the apse, and light behavior in the original dome. Our findings point to the potential of digital tools in the discovery of a structure’s hidden features and design rules.
1. Introduction

The Hagia Sophia church was built during the years 532–537 by the Byzantine emperor Justinian and designed by Isidore of Miletus and Anthemius of Tralles—a physicist and a mathematician respectively [1]. Its dedication to Hagia Sophia, did not refer to any saint by that name but to Christ as the Wisdom (Sophia) or Word of God made flesh, which is confirmed by the fact that the patronal feast was celebrated at Christmas, on December 25th [2], [3]. This was the imperial church in which most major celebrations were held [A]. In Justinian times the emperor participated in thirteen major celebrations per year [B]. By the 6th century, it was already considered the eighth wonder of the world [1].

The church experienced several disasters in which parts of it were demolished. In 558, the original dome collapsed and was replaced by a higher dome. The church walls were reinforced and the church was re-dedicated in 563. During later years, the new dome also partially collapsed and required local rebuilding. Several modifications to the interior and exterior considerably altered the behavior of light in the church.

The Hagia Sophia church in Istanbul, Turkey served as an experimental test case in which advanced knowledge of geometrical constructs, sophisticated understanding of light behavior, and religious and cosmological beliefs combined to create a magnificent structure and a monument to knowledge. In this paper, we integrate traditional historical research, parametric digital analysis, and lighting simulation to analyze several aspects of the church. In particular, we focus on the geometry of the floor plan, the geometry of the apse, and light behavior in the original dome.

2. The role of geometry, light, and cosmology

Hagia Sophia’s design seems to be the embodiment of the integration of geometry, light, and cosmology. Procopius, Emperor Justinian’s court historian, account of the church indicates how these three concerns factored into its design:

“… Its breadth and length have been so fittingly proportioned that it may rightly be said to be both very long and unusually broad … [It] abounds in sunlight and gleaming reflections. Indeed one might say that its radiance generated within, such is the abundance of light that bathes this shrine all around … And whenever one enters the church to pray, one understands immediately that it has been fashioned not by any human power or skill but by the influence of God.” [4]
Our research points to these three concerns as the major factors in designing the church. The system of ritual lighting involved the apse and dome design [5, 6], the proportions of the church interior in respect to the way its various spaces ought to be viewed [1, 7], and a number of additional issues [8]. In the next sections we explore the role of geometry, light, and cosmology in three aspects of the church’s design: the overall floor plan, the apse, and the dome. In each part, we found evidence of advanced knowledge of mathematics, light, and cosmology. In particular, the design of Hagia Sophia perfected the use of sophisticated geometric proportions to create a spatial spectacle that admits natural light for the generation of an atmosphere supportive of the liturgical acts.

3. The main floor plan

Structural recordings by Roberts L. Van Nice of the Hagia Sophia church were first published in 1965 and then again in 1986 [9]. In 2005, Volker Hoffmann performed measurements and structural recordings by means of a Cyrax 2005 laser scanner. He overlapped his digital recordings over the Van Nice plans and found them to be extremely accurate [1]. Our analysis here constitutes a further digital verification of Hoffmann’s work via the deployment of a parametric model that follows his description of the system of proportions.

The basic design of Hagia Sophia starts with a square and two circles; one inscribed and one circumscribed. The cosmological reference, Hoffmann contends, is that to the cube and the sphere. The cube symbolizes the order of the universe while the sphere symbolizes its physical form. The square measures 100 Byzantine feet. The figure is then doubled with another square that measures 106 feet and two circles are then inscribed in and circumscribed about the second larger square. This basic geometric figure of doubled squares and circles with a ratio of 100:106 is then used to create two more identical figures that use the largest two circles and the smallest two circles. The resulting figure creates a set of 25 points including the figure’s centroid on which to base other arcs that would be swept to determine the location of walls.

In order to verify the accuracy of this construction, a parametric model was constructed using Bentley’s Generative Components (GC) software. The model allows for the construction of exact geometric relationships that essentially test the validity of the geometric proposition (Figure 1). If a geometric relation does not hold true, then the attempt to model it using the GC software would fail. The results of the verification model are illustrated in Figure 2. The scope of this paper does not allow for a detailed explanation of the exact geometric construction.
4. The apse

This section adapts and extends our earlier research on the design of the Hagia Sophia apse geometry [10]. The apse was designed to admit a light shaft on the altar through one of the apse windows on a significant time of day of important celebration dates. According to Christian dogma, the significant time was the Byzantine third hour during which the Holy Ghost descended to sanctify the offerings presented to God by the priest in front of the altar [C]. This central liturgical rite necessitated the incidence of a light shaft on the offerings resembling the descent of the Holy Spirit. In order for this to occur, the church orientation was determined by the Byzantine timekeeping method. This method was based on the assumption that both day and night were subdivided into twelve hours each. But since the day, for instance, varied in duration according to the season, the hours varied likewise becoming longer in the summer and shorter in the winter. The third hour corresponded always to the middle of the morning but when compared to our timekeeping method it appears to vary according to the season [11].
According to our observations, in designing the church apse a circle was established first which determined both the interior side of the round apse wall and the location of the altar. The circle center seems to have been placed on the junction of the church’s longitudinal axis with the interior side of the eastern wall of the main building. The location of the altar was determined by this circle but the exact position of the priest was determined by a polygon circumscribed about it. A second larger concentric circle was drawn, to account for the thickness of the wall, about which another similar polygon was circumscribed. The method of inscribing and circumscribing shapes is consistent with the overall design methodologies as seen in the above section. In Byzantine churches, regular polygons were usually used. However, the number of their sides could vary considerably from one church to another. We did not find any written reference to the geometrical shape of the apse of the Hagia Sophia. The most detailed description states that the apse is semi-circular on the inside and three-sided.
Therefore, we set out to discover its exact geometric shape based on our analysis of the liturgical requirements, an analysis of sunlight, the angles of the walls, and the mathematical and cosmological context of the architects who designed it.

Usually the type of polygon used was thought to be a matter of stylistic preference while we maintain that to a certain extent it was determined by the intention to direct visible light shafts toward the altar in the manner outlined above. In order to make a decision on the type of polygon to be used probably two or more important celebration dates had to be set and the corresponding direction of sunlight be established for each.

We found that the axis of the church is precisely aligned with the third hour of the equinox with an azimuth angle of 123.4° [D]. We then used a specialized computer program [E] to verify that our observations correspond to the alignments of sun azimuth angles with the altar on the particular dates and hours of the year of its design (532).

A major finding of our analysis is that for the apse design the aforementioned basic circle appears to have been used along with a circumscribed regular heptagon (7-sided polygon) about it with one of its chords set parallel to the exterior wall and the opposite apex located precisely in front of the altar. The heptagon has an angle at center corresponding to the chord which is \( \frac{360°}{7} = 51.4° \). The heptagon apex at the altar aligns on December 25th of 532, with the sun azimuth of the third hour 146.9° through the axis of the southernmost window of the apse [F].

The angle of the direction of light on equinox and Christmas days of 532 most closely approximate the inscribed angle of a heptagon. One might think that an even-sided polygon would be chosen (e.g. an octagon or a hexagon). The difference is not so great as to justify the selection of the heptagon over an octagon [10]. Apart from the heptagon’s advantage in respect to the determination of the position of the priest by an apex there might be additional reasons which might be stemming from the symbolic significance of the heptagon. The heptagon carries multiple symbolic connotations among which is its association with Virgin Mary, since the number 7 is thought of as the virgin number. It also corresponds to the then thought of as the 7 planets of the planetary system which included the 5 planets, the sun and the moon. Furthermore, it stands for the fusion of divine and human since number three, symbolizing divine nature, combines with four, symbolizing human nature.

Geometrically, a regular heptagon is considered to be impossible to construct by legal use of a compass and a straightedge. Instead, Greek geometers knew about a construction method invented by Archimedes that they termed a Neusis-construction [12]. The method included marking a straight edge and rotating it about a point until it meets another edge and marking that as an initial vertex of the heptagon. A more detailed description of a neusis-construction can be found in [12]. In order to verify Archimedes’s construction, a parametric model was created using...
the provided description and animated exactly as described. Indeed, the model created a regular heptagon when the specified conditions were met (Figure 3). It is not unreasonable to conclude that the construction of the chords of a regular heptagon showed-off the prowess of a mathematician such as Anthemius of Tralles.

The parametric model for the Hagia Sophia apse followed the above description of the geometric relationships. Yet, it allowed the user to input the non-dependent values such as the radius of the apse, the wall thickness, the azimuth angle of the church axis, and the azimuth angle of the incident sun ray. A script iterated through the possible number of polygons and each was presented to the user. An embedded script also analyzed the polygon to suggest an ’ideal’ shape given an evaluation function (Figure 4).

In order to verify that the geometry of the apse was indeed based on a heptagon, the resulting drawing from the parametric drawing was placed over the plan of the church apse (Figure 5). The exactness of the match between the two drawings convinced us that indeed a heptagon was used as the geometrical figure of the apse. Again, we see here the combination of an advanced knowledge of geometry, light behavior, and cosmological and religious beliefs in the construction of a feature of the church.

5. The dome
This section summarizes, adapts and extends our earlier work on the behavior of light in the Hagia Sophia dome [6]. Byzantine church domes, even today, often appear to emit light from within. The radiance of the dome is enhanced
Figure 4: A still image from an interactive parametric model for the creation of a regular geometry for a church apse.

Figure 5: A heptagon superimposed over the plan of the apse of Hagia Sophia.
by the shape of the pendentives that are also lit and appear to dynamically lift the dome off the ground (Figure 6). It imparts the impression of a luminous complex launched from four points at the base and stretching over the interior space as if it were a weightless membrane not subjected to the law of earthly gravity, a notion carrying clear symbolic connotations. The image of Pantocrator, often placed at the apex, is made to appear to emit constant light.

Our understanding of the behavior of light in Hagia Sophia can be enhanced by analyzing a study of two reflectors undertaken by Anthemius of Tralles. One of these reflectors was designed for the purpose of directing solar light at a single point within a building, taking into account the varying positions of the sun throughout the day and the seasons (Figure 7). The geometrical solution to this problem appears in his excerpt entitled *Peri Paradwse Michanimaton* (About Paradoxical Machines) reproduced and with an English translation in [13].

As it is revealed by these written sources, Anthemius was an expert in the handling of light, being able to direct, focus, and stabilize it. Anthemius states: “It is required to cause a ray of the sun to fall in a given position, without moving away, at any hour or season.” While we have no written proof that Anthemius used this reflector for Hagia Sophia, the concern with the seasons and shining a light on a specific point is consistent with a concern for the liturgical requirements of a church. It should be expected, thereby, that he would use his capacities and knowledge of light and the movement of the sun in the sky in designing Hagia Sophia. The manner of the admission of light, as it appears, was critical to the new church, a fact which not only led Anthemius to creating reflectors of local importance but, as we shall see, played a significant role in the shaping of major formal elements as well—in particular the shape of the dome and pendentives.

Regarding this particular reflector, there are clear indications both in the drawing and the accompanying text that it was intended for Hagia Sophia.
These indications have been thoroughly analyzed in [8] and [5]. In order to test the validity of this hypothesis, we analyzed both the geometry of the proposed reflector and the behavior of light reflections in it.

First, however, in order to verify that Anthemius’s reflector (Figure 7) actually works, his textual instructions were followed to create a parametric model using Bentley’s Generative Components (GC) software. We used GC to construct the mechanics of Anthemius’s reflector with the ability to modify the angles and the distances at will. The model behaved as expected and does verify that light passing through point B will indeed be reflected to point A (Figure 8).
This prompted us to propose that this profile may have been a possible solution for a surface for the parapet at the bottom of the dome. Our proposal makes use of a 3D curved ellipsoidal surface that would direct the light to the apex of the dome regardless of the angle of the incident light ray (Figure 9). Later in this section, we model the behavior of light and present evidence that an ellipsoidal parapet coupled with Anthemius’s second reflector, as discussed below, combined to create the gleam as described by writings of the time.

In continuation of our research of Anthemius’s reflectors, we constructed a second parametric model that analyzed the reflection of light rays as they would enter the dome. We discovered that rays of light entering a shallower dome from below (as would be reflected from the floor of the church as well as more oblique angles (as would enter from side walls) would inter-reflect within a shallower dome to create an even more radiant image (Figure 10).

Furthermore, in Anthemius’s design for a second reflector, we found that the curvature profile of the original dome in transverse section is identical with the arc shape described by Anthemius’ design for the reflector [G]. It is especially remarkable that the base of the inscribed square coincides exactly with the lower cornice while the upper horizontal line of the square coincides precisely with the cornice at the base of the dome windows (Figure 11).

A lighting simulation using accurate lighting techniques provided further evidence that the design of the dome and pendentives was the result of careful manipulation of light in order to achieve a liturgical and cosmological effect. Computer-based lighting models have been shown to be accurate.
tools [14]. In our study, we used AutoDesk’s 3D Studio MAX and applied the most accurate advanced lighting method (Radiosity combined with ray-tracing with high-accuracy settings). We constructed the geometry of the dome and two versions of the parapet: the one proposed by Mainstone and the one proposed by us. We then rendered each, viewing the dome from the bottom looking up at the dome (Figure 12). Using Adobe Photoshop, we applied the same curve distortion to both original images to better discern the light distribution in the dome (Figure 13). It is clear that the ellipsoidal reflector created a brighter and more uniform light distribution. The geometry of the curved mirror and the flat geometry of the dome combined to create an effect of an “incessant gleam” as suggested by accounts from
that period.

Then, we added the pendentives to the 3D model which revealed that the light reflections would illuminate them very evenly and more so at their base increasing the impression of weightlessness (Figure 14). The lighting curve distortion clearly reveals the high level of evenness in their illumination despite the strong directionality of sunlight (Figure 15).

6. Conclusion

Geometry, light and cosmology were masterfully combined by the architects of Hagia Sophia to create an exquisitely proportioned religious space bathed in light. The availability of sophisticated parametric digital tools and advanced lighting models has allowed us to test many of the theoretical propositions and discover new findings regarding the attributes of some of its architectural components—the floor plan, apse, and dome. In all three cases, we found that the design was derived from the integration of geometry, light, and cosmology. The parametric models allowed us to analyze and verify
Anthemius’ research on advanced geometrical constructs, such as the construction method for a heptagon, and reflecting mirrors that manipulate the light with accuracy. On the other hand, accurate lighting models provided visualizations of the lighting conditions and further verified the historical descriptions of the church at the time it was built. Finally, when comparing the analysis with historical research, we can then understand how strong religious and cosmological beliefs have been symbolized through the use of geometry and the control of light.

Notes

A. Such as Christmas (Dec. 25th), Epiphany (Jan. 6th), Good Friday, Easter, Monday after Easter, Pentecost, Exaltation of the Cross (Sept. 14th)
etc. Major feasts were also the Annunciation (March 25th), Circumcision (Jan. 1st), Transfiguration (Aug. 6th), the anniversary of the foundation of the city (May 11th), and Constantine's and Helen's feast (May 21st) [3], pp. 16, 115.

B. The Book of Ceremonies cited in [2], p. 231, states that the emperor participated in the liturgy on seventeen occasions a year. However, four of them were feasts introduced after the time of Justinian.

C. According to the New Testament the Holy Ghost descended on the third hour of Pentecost in the form of light to enlighten the Apostles and enable them to teach the Christian faith. Acts of the Apostles b, 1-15. Correspondingly on the third hour, during the Eucharist celebration, the priest prays to God to send the Holy Ghost upon the offerings in order to transform them into the Body and Blood of Christ which, then, can similarly enlighten those who receive them [3], pp. 114, 127.

D. This azimuth angle is published by Van Nice (1965), whose work is the most precise survey available of the church to date. See [9].

E. Program code by Jeffrey Sax © 1991-1992 distributed by Willman-Bell, Inc. no. 33854 as elaborated on by Associate Professor Richard Sears of the University of Michigan.

F. Additional dates were served as well but due to shortage of space this cannot be dealt with here.

G. See [15], p. 589 and [16], p. 37.

References


Wassim Jabi
New Jersey Institute of Technology
School of Architecture
Newark, NJ 07102-1982 USA
wassim.jabi@gmail.com

Iakovos Potamianos
Democritus University of Thrace
Department of Architecture
Xanthi, Greece
ipota@tee.gr
Multimodal Mixed Reality Interfaces for Visualizing Digital Heritage

Martin White, Panagiotis Petridis, Fotis Liarokapis and Daniel Plecinckx